MECHANICS FOR ENGINEERS

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A TEXT-BOOK OF INTERMEDIATE STANDARD

BY

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WITH 200 DIAGRAMS AND NUMEROUS EXAMPLES

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PREFACE

ENGINEERING students constitute a fairly large proportion of those attending the Mechanics classes in technical colleges and schools, but their needs are not identical with those of the students of general science. It has recently become a common practice to provide separate classes in Mathematics, adapted to the special needs of engineering students, who are in most institutions sufficiently numerous to justify similar provision in Mechanics. The aim of this book is to provide a suitable course in the principles of Mechanics for engineering students.

With this object in view, the gravitational system of units has been adopted in the English measures. A serious injustice is often done to this system in books on Mechanics by wrongly defining the pound unit of force as a variable quantity, thereby reducing the system to an irrational one. With proper premises the gravitational system is just as rational as that in which the "poundal" is adopted as the unit of force, whilst it may be pointed out that the use of the latter system is practically confined to certain text-books and examination papers, and does not enter into any engineering work. Teachers of Engineering often find that students who are learning Mechanics by use of the "poundal" system, fail to apply the principles to engineering problems stated in the only units which are used in such cases—the gravitational

units. The use of the dual system is certainly confusing to the student, and in addition necessitates much time being spent on the re-explanation of principles, which might otherwise be devoted to more technical work.

Graphical methods of solving problems have in some cases been used, by drawing vectors to scale, and by estimating slopes and areas under curves. It is believed that such exercises, although often taking more time to work than the easy arithmetic ones which are specially framed to give exact numerical answers, compel the student to think of the relations between the quantities involved, instead of merely performing operations by fixed rules, and that the principles so illustrated are more deeply impressed.

The aim has not been to treat a wide range of academic problems, but rather to select a course through which the student may work in a reasonable time—say a year—and the principles have been illustrated, so far as the exclusion of technical knowledge and terms would allow, by examples likely to be most useful to the engineer.

In view of the applications of Mechanics to Engineering, more prominence than usual has been given to such parts of the subject as energy, work of forces and torques, power, and graphical statics, while some other parts have received less attention or have been omitted.

It is usual, in books on Mechanics, to devote a chapter to the equilibrium of simple machines, the frictional forces in them being considered negligible: this assumption is so far from the truth in actual machines as to create a false impression, and as the subject is very simple when treated experimentally, it is left for consideration in lectures on Applied Mechanics and in mechanical laboratories.

The calculus has not been used in this book, but the

student is not advised to try to avoid it; if he learns the elements of Mechanics before the calculus, dynamical illustrations of differentiation and integration are most helpful. It is assumed that the reader is acquainted with algebra to the progressions, the elements of trigonometry and curve plotting; in many cases he will doubtless, also, though not necessarily, have some little previous knowledge of Mechanics.

The ground covered is that required for the Intermediate (Engineering) Examination of the University of London in Mechanics, and this includes a portion of the work necessary for the Mechanics Examination for the Associateship of the Institution of Civil Engineers and for the Board of Education Examination in Applied Mechanics.

I wish to thank Professor W. Robinson, M.E., and Professor J. Goodman for several valuable suggestions made with respect to the preparation and publication of this book; also Mr. G. A. Tomlinson, B.Sc., for much assistance in correcting proofs and checking examples; in spite of his careful corrections some errors may remain, and for any intimation of these I shall be obliged.

ARTHUR MORLEY.

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CONTENTS

CHAPTER I

KINEMATICS

CHAPTER II

THE LAWS OF MOTION

CHAPTER III

WORK, POWER, AND ENERGY

CHAPTER IV

MOTION IN A CIRCLE: SIMPLE HARMONIC MOTION

CHAPTER V

STATICS—CONCURRENT FORCES—FRICTION

CHAPTER VI

STATICS OF RIGID BODIES

CHAPTER VII

CENTRE OF INERTIA OR MASS—CENTRE OF GRAVITY

CHAPTER VIII

CENTRE OF GRAVITY—PROPERTIES AND APPLICATIONS

CHAPTER IX

MOMENTS OF INERTIA—ROTATION

CHAPTER X

ELEMENTS OF GRAPHICAL STATICS

MECHANICS FOR ENGINEERS

CHAPTER I

KINEMATICS

1. Kinematics deals with the motion of bodies without reference to the forces causing motion.

Motion in a Straight Line.

Velocity.—The velocity of a moving point is the rate of change of its position.

Uniform Velocity.—When a point passes over equal spaces in equal times, it is said to have a *constant velocity*; the magnitude is then specified by the number of units of length traversed in unit time, *e.g.* if a stone moves 15 feet with a constant velocity in five seconds, its velocity is 3 feet per second.

If s = units of space described with constant velocity v in t units of time, then, since v units are described in each second, $(v \times t)$ units will be described in t seconds, so that—

$$s = vt$$

$$\text{and} \quad v = \frac{s}{t}$$

Fig. 1 shows graphically the relation between the space described and the time taken, for a constant velocity of 3 feet per second. Note that $v = \frac{s}{t} = \frac{9}{3}$ or $\frac{6}{2}$ or $\frac{3}{1}$, a constant

velocity of 3 feet per second whatever interval of time is considered.

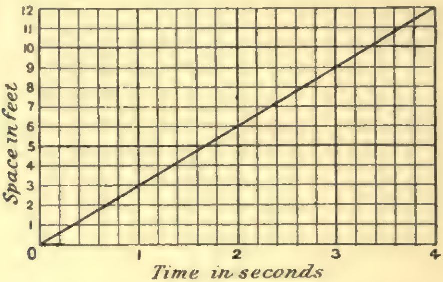


FIG. 1.—Space curve for a uniform velocity of 3 feet per second.

2. Mean Velocity.—The mean or average velocity of a point in motion is the number of units of length described, divided by the number of units of time taken.

3. Varying Velocity.—The actual velocity of a moving point at any instant is the mean velocity during an indefinitely small interval of time including that instant.

4. The Curve of Spaces or Displacements.—Fig. 2 shows graphically the relation between the space described and

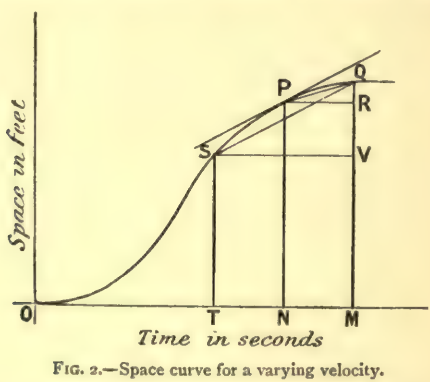


FIG. 2.—Space curve for a varying velocity.

the time taken for the case of a body moving with a varying velocity. At a time ON the displacement is represented by

PN, and after an interval NM it has increased by an amount QR, to QM. Therefore the *mean velocity* during the interval NM is represented by $\frac{QR}{NM}$ or $\frac{QR}{PR}$ or by tan $Q\hat{P}R$, *i.e.* by the tangent of the angle which the chord PQ makes with a horizontal line. If the interval of time NM be reduced indefinitely, the chord PQ becomes the tangent line at P, and the mean velocity becomes the velocity at the time ON. Hence *the velocity at any instant is represented by the gradient of the tangent line to the displacement curve at that instant.* An upward slope will represent a velocity in one direction, and a downward slope a velocity in the opposite direction.

5. If the curvature is not great, *i.e.* if the curve does not bend sharply, the best way to find the direction of the tangent line at any point P on a curve such as Fig. 2, is to take two ordinates, QM and ST, at short equal distances from PN, and join QS; then the slope of QS, viz. $\frac{QV}{SV}$, is approximately the same as that of the tangent at P. This is equivalent to taking the velocity at P, which corresponds to the middle of the interval TM, as equal to the mean velocity during the interval of time TM.

6. Scale of the Diagram.—Measure the slope as the gradient or ratio of the vertical height, say QV, to the horizontal SV or TM. Let the ratio QV : TM (both being measured in inches say) be x. Then to determine the velocity represented, note the velocity corresponding to a slope of 1 inch vertical to 1 inch horizontal, say y feet per second. Then the slope of QS denotes a velocity of xy feet per second.

7. Acceleration.—The acceleration of a moving body is the rate of change of its velocity. When the velocity is increasing the acceleration is reckoned as positive, and when decreasing as negative. A negative acceleration is also called a retardation.

8. Uniform Acceleration.—When the velocity of a point increases by equal amounts in equal times, the acceleration is said to be *uniform or constant:* the magnitude is then specified

by the number of units of velocity per unit of time; *e.g.* if a point has at a certain instant a velocity of 3 feet per second, and after an interval of eight seconds its velocity is 19 feet per second, and the acceleration has been uniform, its magnitude is $\frac{\text{increase of velocity}}{\text{time taken to increase}} = \frac{19-3}{8} = 2$ feet per second in each of the eight seconds, *i.e.* 2 feet per second *per second*. At the end of the first, second, and third seconds its velocities would be $(3+2)$, $(3+4)$, and $(3+6)$ feet per second respectively (see Fig. 3).

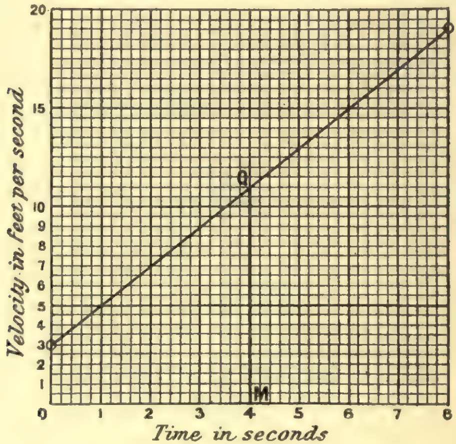


FIG. 3.—Uniform acceleration.

9. Mean Acceleration.—The acceleration from 3 feet per second to 19 feet per second in the last article was supposed uniform, 2 feet per second being added to the velocity in each second; but if the acceleration is variable, and the increase of velocity in different seconds is of different amounts, then the acceleration of 2 feet per second per second during the eight seconds is merely the *mean acceleration* during that time. The mean acceleration is equal to $\frac{\text{increase of velocity}}{\text{time taken for increase}}$, and is in the direction of the change of velocity.

The **actual acceleration** at any instant is the mean acceleration for an indefinitely small time including that instant.

10. Fig. 3 shows the curve of velocity at every instant during the eight seconds, during which a point is uniformly accelerated from a velocity of 3 feet per second to one of 19 feet per second.

11. Calculations involving Uniform Acceleration.—If u = velocity of a point at a particular instant, and f = uniform acceleration, *i.e.* f units of velocity are added every second—

then after	1 second	the velocity	will be	$u + f$
and ,,	2 seconds	,,	,,	$u + 2f$
,, ,,	3 ,,	,,	,,	$u + 3f$
,, ,,	t ,,	,,	v will be	$u + ft$ (1)

e.g. in the case of the body uniformly accelerated 2 feet per second per second from a velocity of 3 feet per second to a velocity of 19 feet per second in eight seconds (as in Art. 8), the velocity after four seconds is $3 + (2 \times 4) = 11$ feet per second.

The space described (s) in t seconds may be found as follows: The initial velocity being u, and the final velocity being v, and the change being uniform, the mean or average velocity is $\frac{u + v}{2}$.

$$\text{Mean velocity} = \frac{u + v}{2} = \frac{u}{2} + \frac{u + ft}{2} = u + \tfrac{1}{2}ft$$

(which is represented by QM in Fig. 3. See also Art. 2).

$$\text{Hence } u + \tfrac{1}{2}ft = \frac{s}{t}$$

$$\text{and } s = ut + \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad (2)$$

e.g. in the above numerical case the mean velocity would be—

$$\frac{3 + 19}{2} = 11 \text{ feet per second (QM in Fig. 3)}$$

$$\text{and } s = 11 \times 8 = 88 \text{ feet}$$

$$\text{or } s = 3 \times 8 + \tfrac{1}{2} \times 2 \times 8^2 = 24 + 64 = 88 \text{ feet}$$

It is sometimes convenient to find the final velocity in

terms of the initial velocity, the acceleration, and the space described. We have—

from (1) $v = u + ft$

therefore $v^2 = u^2 + 2uft + f^2t^2 = u^2 + 2f(ut + \frac{1}{2}ft^2)$

and substituting for $(ut + \frac{1}{2}ft^2)$ its value s from (2), we have—

$$v^2 = u^2 + 2fs \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The formulæ (1), (2), and (3) are useful in the solution of numerical problems on uniformly accelerated motion.

12. Acceleration of Falling Bodies.—It is found that bodies falling to the earth (through distances which are small compared to the radius of the earth), and entirely unresisted, increase their velocity by about 32·2 feet per second every second during their fall. The value of this acceleration varies a little at different parts of the earth's surface, being greater at places nearer to the centre of the earth, such as high latitudes, and less in equatorial regions. The value of the "acceleration of gravity" is generally denoted by the letter g. In foot and second units its value in London is about 32·19, and in centimetre and second units its value is about 981 units.

13. Calculations on Vertical Motion.—A body projected vertically downwards with an initial velocity u will in t seconds attain a velocity $u + gt$, and describe a space $ut + \frac{1}{2}gt^2$.

In the case of a body projected vertically upward with a velocity u, the velocity after t seconds will be $u - gt$, and will be upwards if gt is less than u, but downward if gt is greater than u. When t is of such a value that $gt = u$, the downward acceleration will have just overcome the upward velocity, and the body will be for an instant at rest: the value of t will then be $\frac{u}{g}$. The space described upward after t seconds will be $ut - \frac{1}{2}ft^2$.

The time taken to rise h feet will be given by the equation—

$$h = ut - \tfrac{1}{2}ft^2$$

This quadratic equation will generally have two roots, the

smaller being the time taken to pass through h feet upward, and the larger being the time taken until it passes the same point on its way downward under the influence of gravitation.

The velocity v, after falling through "h" feet from the point of projection downwards with a velocity u, is given by the expression $v^2 = u^2 + 2gh$, and if $u = 0$, *i.e.* if the body be simply dropped from rest, $v^2 = 2gh$, and $v = \sqrt{2gh}$ after falling h feet.

14. Properties of the Curve of Velocities.—Fig. 4 shows the velocities at all times in a particular case of a body

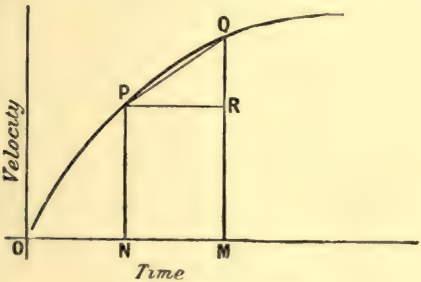


FIG. 4.—Varying velocity.

starting from rest and moving with a varying velocity, the acceleration not being uniform.

(1) **Slope of the Curve.**—At a time ON the velocity is PN, and after an interval NM it has increased by an amount QR to QM; therefore the mean acceleration during the interval NM is represented by $\frac{QR}{NM}$ or $\frac{QR}{PR}$, *i.e.* by the tangent of the angle which the chord PQ makes with a horizontal line. If the interval of time NM be reduced indefinitely, the chord PQ becomes the tangent line to the curve at P, and the mean acceleration becomes the acceleration at the time ON. *So that the acceleration at any instant is represented by the gradient of the tangent line at that instant.* The slope will be upward if the velocity is increasing, downward if it is decreasing; in the latter case the gradient is negative. The scale of accelerations is easily found by the acceleration represented by unit gradient. If the curve does not bend sharply, the direction of the

tangent may be found by the method of Art. 5, which is in this case equivalent to taking the acceleration at P as equal to the mean acceleration during a small interval of which PN is the velocity at the middle instant.

(2) **The Area under the Curve.**—If the velocity is constant and represented by PN (Fig. 5), then the distance described in an interval NM is PN.NM, and therefore the area under PQ, viz. the rectangle PQMN, represents the space described in the interval NM.

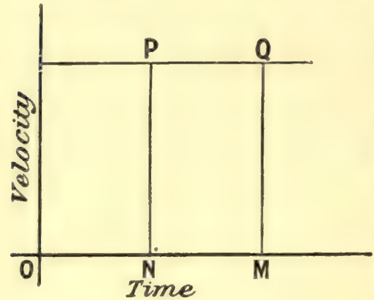


Fig. 5.

If the velocity is not constant, as in Fig. 6, suppose the interval NM divided up into a number of small parts such as CD. Then AC represents the velocity at the time represented by OC; the velocity is increasing, and therefore in the interval CD the space described is greater than that represented by the rectangle AEDC, and less than that represented by the rectangle FBDC. The total space described during the interval

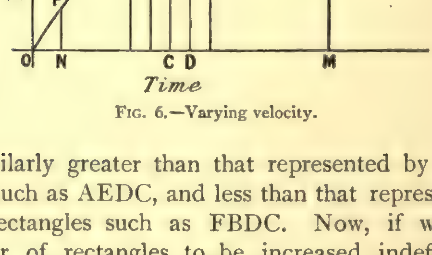


Fig. 6.—Varying velocity.

NM is similarly greater than that represented by a series of rectangles such as AEDC, and less than that represented by a series of rectangles such as FBDC. Now, if we consider the number of rectangles to be increased indefinitely, and

the width of each to be decreased indefinitely, the area PQMN under the curve PQ is the area which lies always between the sums of the areas of the two series of rectangles, however nearly equal they may be made by subdividing NM, and the area PQMN under the curve therefore represents the space described in the interval NM.

The area under the curve is specially simple in the case of uniform acceleration, for which the curve of velocities is a straight line (Fig. 7). Here the velocity PN being u, and NM being t units of time, and the final velocity being QM $= v$, the area under PQ is—

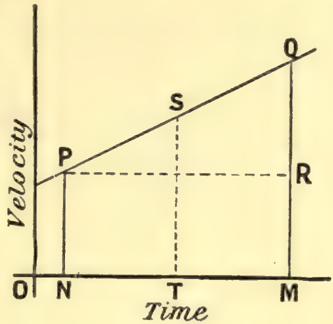


Fig. 7.

$$\frac{\text{PN} + \text{QM}}{2} \times \text{NM} = \text{ST} \times \text{NM}$$

$$\text{or } \frac{u + v}{2} \times t \text{ (as in Art. 11)}$$

And if f is the acceleration $f = \frac{v - u}{t}$ (represented by $\frac{\text{QR}}{\text{NM}}$ or $\frac{\text{QR}}{\text{PR}}$, *i.e.* by tan $\text{Q}\hat{\text{P}}\text{R}$),

$$\therefore ft = v - u$$
$$v = u + ft$$

and the space described $\frac{u + v}{2} \times t$ is $\frac{u + u + ft}{2} \times t$, which is $ut + \frac{1}{2}ft^2$ (as in Art. 11).

15. Notes on Scales.—If the scale of velocity is 1 inch to x feet per second, and the scale of time is 1 inch to y seconds, then the area under the curve will represent the distance described on such a scale that 1 square inch represents xy feet.

16. In a similar way we may show that the area PQMN

(Fig. 8) under a curve of accelerations represents the total increase in velocity in the interval of time NM.

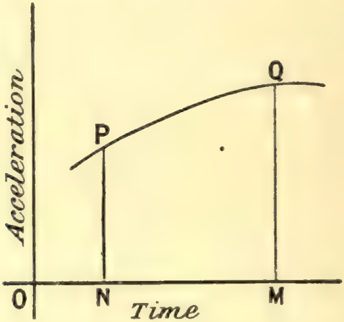


FIG. 8.

If the scale of acceleration is 1 inch to z feet per second per second, and the scale of time is 1 inch to y seconds, then the scale of velocity is 1 square inch to yz feet per second.

17. Solution of Problems.—Where the motion is of a simple kind, such as a uniform velocity or uniform acceleration, direct calculation is usually the easiest and quickest mode of solution, but where (as is quite usual in practice) the motion is much more complex and does not admit of simple mathematical expression as a function of the time taken or distance covered, a graphical method is recommended. Squared paper saves much time in plotting curves for graphical solutions.

Example 1.—A car starting from rest has velocities v feet per second after t seconds from starting, as given in the following table :—

t	0	4	9	17	24	30	32	40	45	53	58	62
v	0	11·0	22·6	35·6	44·5	49·0	48·9	40·6	33 7	26·8	24·3	24·0

Find the accelerations at all times during the first 60 seconds, and draw a curve showing the accelerations during this time.

First plot the curve of velocities on squared paper from the given data, choosing suitable scales. This has been done in

Fig. 9, curve I., the scales being 1 inch to 10 seconds and 1 inch to 20 feet per second.

In the first 10 seconds RQ represents 24·2 feet per second

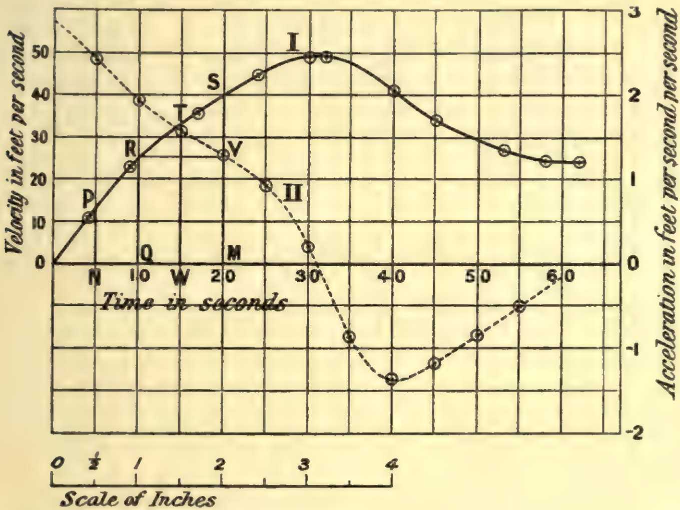


FIG. 9.

gain of velocity, and OQ represents 10 seconds; therefore the acceleration at N 5 seconds from starting is approximately $\frac{24·2}{10}$, or 2·42 feet per second per second. Or thus: unit gradient 1 inch vertical in 1 inch horizontal represents—

$$\frac{\text{20 feet per second}}{\text{10 seconds}} = \text{2 feet per second per second}$$

$$\text{hence } \frac{RQ}{OQ} = \frac{\text{1·21 inch}}{\text{1 inch}} = 1·21$$

$$\left.\begin{array}{l}\text{hence acceleration at N} \\ \text{is } 1·21 \times 2\end{array}\right\} = \text{2·42 feet per second per second}$$

(see Art. 14)

Similarly in the second 10 seconds, SV which is SM − RQ, represents (39·8 − 24·2), or 15·6 feet per second gain of velocity;

therefore the velocity at W 15 seconds from starting is approximately $\frac{15 \cdot 6}{10}$, or 1·56 feet per second per second.

Continue in this way, finding the acceleration at say 5, 15, 25, 35, 45, and 55 seconds from starting; and if greater accuracy is desired, at 10, 20, 30, 40, 50, and 60 seconds also. The simplest way is to read off from the curve I. velocities in tabular form, and by subtraction find the increase, say, in 10 seconds, thus—

t	0	5	10	15	20	25	30	35	40	45	50	55	60
v	0	13·5	24·2	32·8	39·8	45·4	49·0	47·5	40·6	33·7	29·0	25·1	24·1
Change in v for 10 secs.		24·2	19·3	15·6	12·6	9·2	2·1	−8·4	−13·8	−11·6	−8·6	−5·1	
Acceleration		2·42	1·93	1·56	1·26	0·92	0·21	−0·84	−1·38	−1·16	−0·86	−0·51	

From the last line in this table curve II., Fig. 9, has been plotted, and the acceleration at any instant can be read off from it.

It will be found that the area under curve II. from the start to any vertical ordinate is proportional to the corresponding ordinate of curve I. (see Art. 16). The area, when below the time base-line, must be reckoned as negative.

Example 2.—Find the distance covered from the starting-point by the car in Example 1 at all times during the first 60 seconds, and the average velocity throughout this time.

In the first 10 seconds the distance covered is found approximately by multiplying the velocity after 5 seconds by the time, *i.e.* $13 \cdot 5 \times 10 = 135$ feet. This approximation is equivalent to taking 13·5 feet per second as the mean velocity in the first 10 seconds.

In the next 10 seconds the mean velocity being approximately 32·8 feet per second (corresponding to $t = 15$ seconds), the distance covered is $32 \cdot 8 \times 10 = 328$ feet, therefore the total distance covered in the first 20 seconds is $135 + 328 = 463$ feet. Proceeding in this way, taking 10-second intervals throughout the 60 seconds, and using the tabulated results in Example 1, we get the following results:—

t	0	10	20	30	40	50	60
Space in previous 10 secs.	0	135	328	454	475	337	251
Total space	0	135	463	917	1392	1729	1980

from which the curve of displacements, Fig. 10, has been plotted.

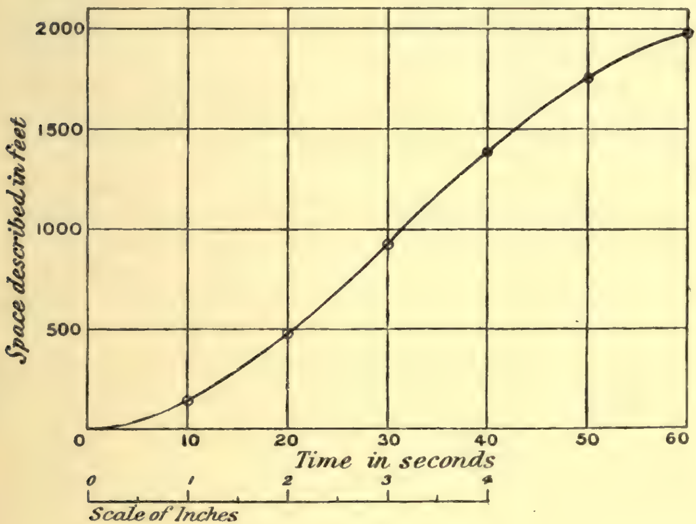


FIG. 10.

Greater accuracy may be obtained by finding the space described every 5 instead of every 10 seconds.

The average velocity $= \frac{\text{space described}}{\text{time taken}} = \frac{1980}{60} = 33{\cdot}0$ feet per sec.

Note that this would be represented on Fig. 9 by a height which is equal to the total area under curve I. divided by the length of base to 60 seconds.

EXAMPLES I.

1. A train attains a speed of 50 miles per hour in 4 minutes after starting from rest. Find the mean acceleration in foot and second units.

2. A motor car, moving at 30 miles per hour, is subjected to a uniform retardation of 8 feet per second per second by the action of its brakes. How long will it take to come to rest, and how far will it travel during this time?

3. With what velocity must a stream of water be projected vertically upwards in order to reach a height of 80 feet?

4. How long will it take for a stone to drop to the bottom of a well 150 feet deep?

5. A stone is projected vertically upward with a velocity of 170 feet per second. How many feet will it pass over in the third second of its upward flight? At what altitude will it be at the end of the fifth second, and also at the end of the sixth?

6. A stone is projected vertically upward with a velocity of 140 feet per second, and two seconds later another is projected on the same path with an upward velocity of 135 feet per second. When and where will they meet?

7. A stone is dropped from the top of a tower 100 feet high, and at the same instant another is projected upward from the ground. If they meet halfway up the tower, find the velocity of projection of the second stone.

The following Examples are to be worked graphically.

8. A train starting from rest covers the distances s feet in the times t seconds as follows :—

t ...	0	5	11	18	22	27	31	38	46	50
s ...	0	10	54	170	260	390	450	504	550	570

Find the mean velocity during the first 10 seconds, during the first 30 seconds, and during the first 50 seconds. Also find approximately the actual velocity after 5, 15, 25, 35, and 45 seconds from starting-point, and plot a curve showing the velocities at all times.

9. Using the curve of velocities from Example 8, find the acceleration every 5 seconds, and draw the curve of accelerations during the first 40 seconds.

10. A train travelling at 30 miles per hour has steam shut off and brakes applied; its speed after t seconds is shown in the following table :—

t	0	4	12	20	26	35	42	50
v, miles per hour ...	30·0	26·0	21·5	16·7	14·0	10·4	7·7	4·8

Find the retardation in foot and second units at 5-second intervals throughout the whole period, and show the retardation by means of a curve. Read off from the curve the retardation after 7 seconds and after 32 seconds. What distance does the train cover in the first 30 seconds after the brakes are applied?

11. A body is lifted vertically from rest, and is known to have the following accelerations f in feet per second per second after times t seconds :—

t ...	0	0·8	1·9	3·0	3·9	4·8	6·0	6·8	8·0	8·8
f ...	3·0	2·9	2·85	2·60	2·20	1·75	1·36	1·20	1·04	0·97

Find its velocity after each second, and plot a curve showing its velocity at all times until it has been in motion 8 seconds. How far has it moved in the 8 seconds, and how long does it take to rise 12 feet?

VECTORS.

18. Many physical quantities can be adequately expressed by a number denoting so many units, *e.g.* the weight of a body, its temperature, and its value. Such quantities are called *scalar quantities*.

Other quantities cannot be fully represented by a number only, and further information is required, *e.g.* the velocity of a ship or the wind has a definite direction as well as numerical magnitude: quantities of this class are called vector quantities and are very conveniently represented by vectors.

A Vector is a straight line having definite length and direction, but not definite position in space.

19. Addition of Vectors.—To find the sum of two vectors

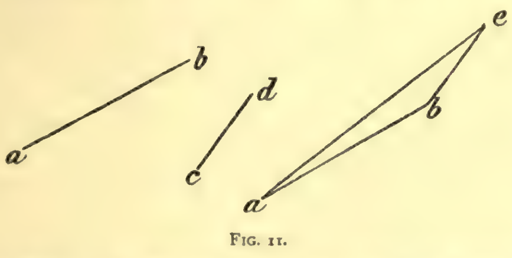


FIG. 11.

ab and *cd* (Fig. 11), set out *ab* of proper length and direction, and from the end *b* set out *be* equal in length and parallel to

cd; join *ae*. Then *ae* is the geometric or vector sum of *ab* and *cd*. We may write this—

$$ab + be = ae$$

or, since *be* is equal to *cd*—

$$ab + cd = ae$$

20. Subtraction of Vectors.—If the vector *cd* (Fig. 12) is to be subtracted from the vector *ab*, we simply find the sum *ae* as before, of a vector *ab* and second vector *be*, which is equal to *cd* in magnitude, but is of opposite sign or direction; then—

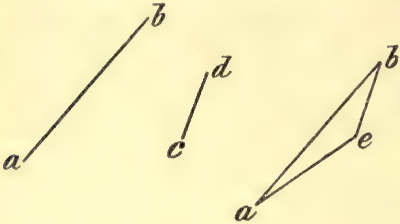


Fig. 12.

$$ae = ab + be = ab - cd$$

If we had required the difference, $cd - ab$, the result would have been *ea* instead of *ae*.

21. Applications: Displacements.—A vector has the two characteristics of a displacement, viz. direction and magnitude, and can, therefore, represent it completely. If a body receives a displacement *ab* (Fig. 11), and then a further displacement completely represented by *cd*, the total displacement is evidently represented by *ae* in magnitude and direction.

22. Relative Displacements. Case I. *Definition.*—If a body remains at rest, and a second body receives a displacement, the first body is said to receive a displacement of equal amount but opposite direction *relative* to the second.

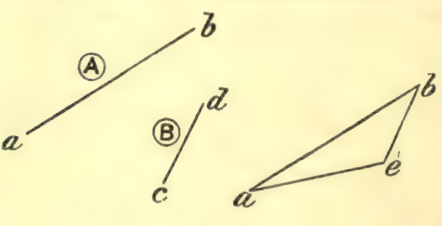


Fig. 13.

Case II. *Where Two Bodies each receive a Displacement.*—If a body A receive a displacement represented by a vector *ab* (Fig. 13), and a body B receive a displacement represented by

cd, then the displacement of A relative to B is the vector difference, $ab - cd$. For if B remained at rest, A would have a displacement *ab* relative to it. But on account of B's motion (*cd*), A has, relative to B, an additional displacement, *dc* (Case I.); therefore the total displacement of A relative to B is $ab + dc$ (or, $ab - cd) = ab + be = ae$ (by Art. 20); where *be* is of equal length and parallel to *dc*.

23. A Velocity which is displacement per unit time can evidently be represented fully by a vector; in direction by the clinure of the vector, and in magnitude by the number of units of length in the vector.

24. Triangle and Polygon of Velocities.—A velocity is said to be the resultant of two others, which are called components, when it is fully represented by a vector which is the geometrical sum of two other vectors representing the two components; *e.g.* if a man walks at a rate of 3 miles per hour across the deck of a steamer going at 6 miles per hour, the resultant velocity with which the man is moving over the sea is the vector sum of 3 and 6 miles per hour taken in the proper directions. If the steamer were heading due north, and the man walking due east, his actual velocity is shown by *ac* in Fig. 14;

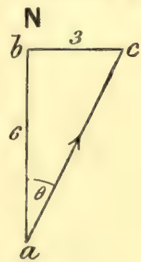


FIG. 14.

$$ab = 6 \qquad\qquad bc = 3$$

$$ac = \sqrt{6^2 + 3^2} = \sqrt{45} \text{ miles per hour}$$
$$= 6{\cdot}71 \text{ miles per hour}$$

and the angle θ which *ac* makes with *ab* E. of N. is given by—

$$\tan\theta = \tfrac{3}{6} = \tfrac{1}{2} \qquad\qquad \theta = 26^\circ\ 35'$$

Resultant velocities may be found by drawing vectors to scale or by the ordinary rules of trigonometry. If the resultant velocity of more than two components (in the same plane) is required, two may be compounded, and then a third with their resultant, and so on, until all the components have been added. It will be seen (Fig. 15) that the result is represented by the closing side of an open polygon the sides of

which are the component vectors. The order in which the sides are drawn is immaterial. It is not an essential condition that all the components should be in the same plane, but if not, the methods of solid geometry should be employed to draw the polygon.

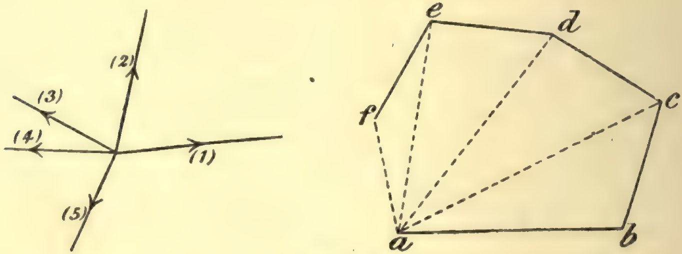


FIG. 15.

Fig. 15 shows the resultant vector *af* of five co-planor vectors, *ab*, *bc*, *cd*, *de*, and *ef*.

$$\text{If, geometrically, } ac = ab + bc$$
$$\text{and } ad = ac + cd$$
$$\text{then } ad = ab + bc + cd$$

and similarly, adding *de* and *ef*—

$$af = ab + bc + cd + de + ef$$

In drawing this polygon it is unnecessary to put in the lines *ac*, *ad*, and *ae*.

25. It is sometimes convenient to resolve a velocity into two components, *i.e.* into two other velocities in particular directions, and such that their vector sum is equal to that velocity.

Rectangular Components.—The most usual plan is to resolve velocities into components in two standard directions at right angles, and in the same plane as the original velocities: thus, if OX and OY (Fig. 16) are the standard directions, and a vector *ab* represents a velocity v, then the component in the direction OX is represented by *ac*, which is equal to *ab* cos θ, and represents v cos θ, and that in the direction OY is represented by *cb*, *i.e.* by *ab* sin θ, and is v sin θ.

This form of resolution of velocities provides an alternative method of finding the resultant of several velocities. Each velocity may be resolved in two standard directions, OX and OY, and then all the X components added algebraically and all the Y components added algebraically. This reduces the components to two at right angles, which may be replaced by a resultant R units, such that

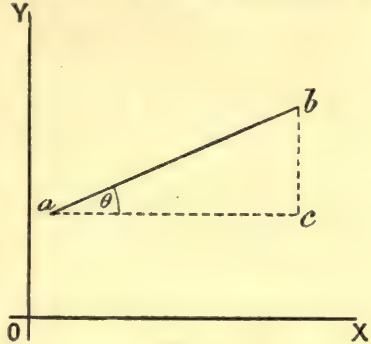


FIG. 16.

the squares of the numerical values of the rectangular components is equal to the square of R, *e.g.* to find the resultant

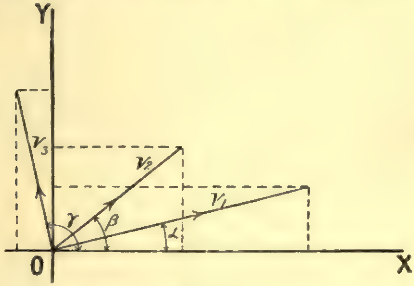


FIG. 17.

of three velocities V_1, V_2, and V_3, making angles α, β, and γ respectively with some fixed direction OX in their plane (Fig. 17).

Resolving along OX, the total X component, say X, is—

$$X = V_1 \cos \alpha + V_2 \cos \beta + V_3 \cos \gamma$$

Resolving along OY—

$$Y = V_1 \sin \alpha + V_2 \sin \beta + V_3 \sin \gamma$$

$$\text{and } R^2 = X^2 + Y^2$$

$$\text{or } R = \sqrt{X^2 + Y^2}$$

and it makes with OX an angle θ such that $\tan \theta = \frac{Y}{X}$.

Fig. 17 merely illustrates the process; no actual drawing of vectors is required, the method being wholly one of calculation.

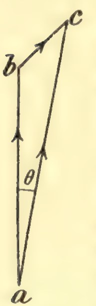


Fig. 18.

Exercise 1.—A steamer is going through the water at 10 knots per hour, and heading due north. The current runs north-west at 3 knots per hour. Find the true velocity of the steamer in magnitude and direction.

(1) By drawing vectors (Fig. 18).

Set off *ab*, representing 10 knots per hour, to scale due north. Then draw *bc* inclined 45° to the direction *ab*, and representing 3 knots per hour to the same scale. Join *ac*. Then *ac*, which scales 12·6 knots per hour when drawn to a large scale, is the true velocity, and the angle $c\hat{a}b$ E of N measures 10°.

(2) Method by resolving N. and E.

$$\text{N. component} = 10 + 3\cos 45^\circ = 10 + \frac{3}{\sqrt{2}} \text{ knots per hour, or } 12{\cdot}12$$

$$\text{E.} \quad ,, \quad = 3\sin 45^\circ = \frac{3}{\sqrt{2}} \text{ knots per hour, or } 2{\cdot}12$$

Resultant velocity $R = \sqrt{(12{\cdot}12)^2 + (2{\cdot}12)^2} = 12{\cdot}6$ knots per hour

And if θ is the angle E. of N. $\left.\right\}$ $\tan\theta = \frac{3}{\sqrt{2}} \div \left(10 + \frac{\sqrt{3}}{2}\right) = \frac{2{\cdot}12}{12{\cdot}12} = 0{\cdot}175$

$$\therefore \theta = 9^\circ\ 55'$$

Relative Velocity.

26. The velocity of a point A relative to a point B is the rate of change of position (or displacement per unit of time) of A with respect to B.

Let v be the velocity of A, and u that of B.

If A remained stationary, its displacement per unit time *relative to* B would be $-u$ (Art. 22). But as A has itself a velocity v, its total velocity relative to B is $v + (-u)$ or $v - u$, the subtraction to be performed geometrically (Art. 20).

The velocity of B relative to A is of course $u - v$, equal in magnitude, but opposite in direction. The subtraction of velocity $v - u$ may be performed by drawing vectors to scale,

by the trigonometrical rules for the solution of triangles, or by the method of Art. 25.

Example.—Two straight railway lines cross: on the first a train 10 miles away from the crossing, and due west of it, is approaching at 50 miles per hour; on the second a train 20 miles away, and 15° E. of N., is approaching at 40 miles per hour. How far from the crossing will each train be when they are nearest together, and how long after they occupied the above positions?

First set out the two lines at the proper angles, as in the left side of Fig. 19, and mark the positions A and B of the first and second

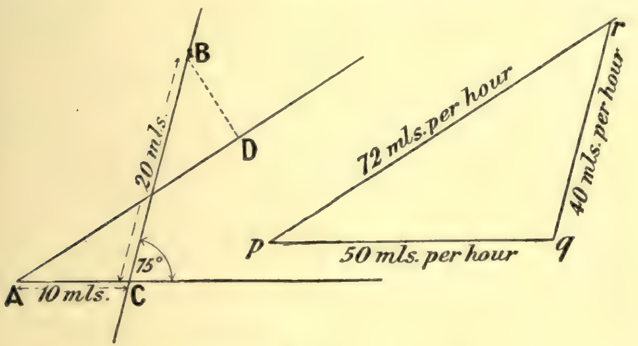


Fig. 19.

trains respectively. Now, since the second train B is coming *from* 15° E. of N., the first train A has, *relative to the second*, a component velocity of 40 miles per hour in a direction E. of N., in addition to a component 50 miles per hour due east. The relative velocity is therefore found by adding the vectors *pq* 50 miles per hour east, and *qr* 40 miles per hour, giving the vector *pr*, which scales 72 miles per hour, and has a direction $57\frac{1}{2}°$ E. of N. Now draw from A a line AD parallel to *pr*. This gives the positions of A relative to B (regarded as stationary). The nearest approach is evidently a distance BD, where BD is perpendicular to AD. The distance moved by A relative to B is then AD, which scales 23·2 miles (the trains being then a distance BD, which scales 8·12 miles apart). The time taken to travel relatively 23·2 miles at 72 miles per hour is $\frac{23 \cdot 2}{72}$ hours = 0·322 hour.

Hence A will have travelled $50 \times 0{\cdot}322$ or $16{\cdot}1$ miles
and B " " $40 \times 0{\cdot}322$ or $12{\cdot}9$ "
A will then be $6{\cdot}1$ miles past the crossing, and
B " " $7{\cdot}1$ " short of the crossing.

27. Composition, Resolution, etc., of Accelerations.—Acceleration being also a vector quantity, the methods of composition, resolution, etc., of velocities given in Arts. 23 to 26 will also apply to acceleration, which is simply velocity added per unit of time. It should be noted that the acceleration of a moving point is not necessarily in the same direction as its velocity: this is only the case when a body moves in a straight line.

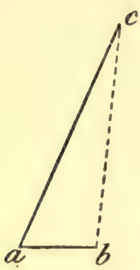


FIG. 20.

If ab (Fig. 20) represents the velocity of a point at a certain instant, and after an interval t seconds its velocity is represented by ac, then the change in velocity in t seconds is bc, for $ab + bc = ac$ (Art. 19), and $bc = ac - ab$ (Art. 20), representing the change in velocity. Then during the t seconds the mean acceleration is represented by $bc \div t$, and is in the direction bc.

28. Motion down a Smooth Inclined Plane.—Let α be the angle of the plane to the horizontal, then the angle $A\hat{B}C$ (Fig. 21) to the vertical is $(90° - \alpha)$. Then, since a body has a downward vertical acceleration g, its component along BA will be $g \cos C\hat{B}A = g \cos (90° - \alpha) = g \sin \alpha$, provided, of course, that there is nothing to cause a retardation in this direction,

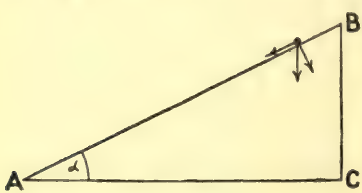


FIG. 21.

i.e. provided that the plane is perfectly smooth and free from obstruction. If BC $= h$ feet, AB $= h$ cosec α feet. The velocity of a body starting from rest at B and sliding down AB will be at A, $\sqrt{2 . g \sin \theta \times h \operatorname{cosec} \theta} = \sqrt{2gh}$, just as if it had fallen h feet vertically.

29. Angular Motion: Angular Displacement.—If P (Fig. 22) be the position of a point, and Q a subsequent position which this point takes up, then the angle $Q\hat{O}P$ is the angular displacement of the point about O. The angular displacement about any other point, such as O′, will generally be a different amount.

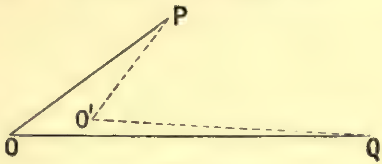


FIG. 22.

30. Angular Velocity.—The angular velocity of a moving point about some fixed point is the rate of angular displacement (or rate of change of angular position) about the fixed point; it is usually expressed in radians per second, and is commonly denoted by the letter ω. As in the case of linear velocity, it may be uniform or varying.

A point is said to have a uniform or constant angular velocity about a point O when it describes equal angles about O in equal times. The mean angular velocity of a moving point about a fixed point O is the angle described divided by the time taken.

If the angular velocity is varying, the actual angular velocity at any instant is the mean angular velocity during an indefinitely small interval including that instant.

31. Angular Acceleration is the rate of change of angular velocity; it is usually measured in radians per second per second.

32. The methods of Arts. 4 to 11 and 14 to 16 are applicable to angular motion as well as to linear motion.

33. To find the angular velocity *about* O of a point describing a circle of radius r about O as centre with constant speed.

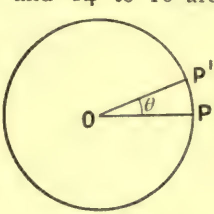


FIG. 23.

Let the path PP′ (Fig. 23) be described by the moving point in t seconds. Let v be the velocity (which, although constant in magnitude, changes direction). Then angular velocity about O is $\omega = \frac{\theta}{t}$.

$$\text{But } \theta = \frac{\text{arc PP}'}{r} \text{ and arc PP}' = vt$$

$$\therefore \theta = \frac{vt}{r} \text{ and } \omega = \frac{\theta}{t} = \frac{vt}{r} \div t = \frac{v}{r}$$

This will still be true if O is moving in a straight line with velocity v as in the case of a rolling wheel, provided that v is the velocity of P *relative* to O.

If we consider t as an indefinitely small time, PP′ will be indefinitely short, but the same will remain true, and we should have $\omega = \frac{v}{r}$ whether the velocity remains constant in magnitude or varies.

In words, the angular velocity is equal to the linear velocity divided by the radius, the units of length being the same in the linear velocity v and the radius r.

Example.—The cranks of a bicycle are $6\frac{1}{2}$ inches long, and the bicycle is so geared that one complete rotation of the crank carries it through a distance equal to the circumference of a wheel 65 inches diameter. When the bicycle is driven at 15 miles per hour, find the absolute velocity of the centre of a pedal—(1) when vertically above the crank axle; (2) when vertically below it; (3) when above the axle and 30° forward of a vertical line through it.

The pedal centre describes a circle of 13 inches diameter relative to the crank axle, *i.e.* 13π inches, while the bicycle travels 65π inches. Hence the velocity of the pedal centre relative to the crank axle is $\frac{1}{5}$ that of the bicycle along the road, or 3 miles per hour.

15 miles per hour = 22 feet per second
3 ,, ,, = 4·4 ,, ,,

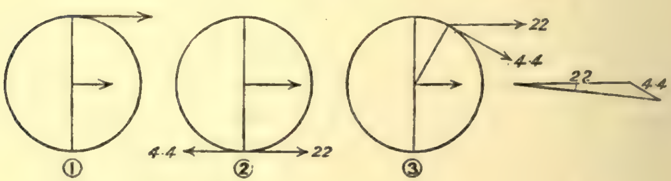


FIG. 24.

(1) When vertically above the crank axle, the velocity of pedal is $22 + 4 \cdot 4 = 26 \cdot 4$ feet per second.

(2) When vertically below the crank axle, the velocity of pedal is $22 - 4 \cdot 4 = 17 \cdot 6$ feet per second.

(3) Horizontal velocity $X = 22 + 4 \cdot 4 \cos 30° = 22 + 2 \cdot 2 \sqrt{3}$ feet per second.

Vertical velocity downwards $Y = 4 \cdot 4 \times \sin 30° = 2 \cdot 2$ feet per second.

Resultant velocity being R—

$$R^2 = (22 + 2 \cdot 2\sqrt{3})^2 + \left(\frac{22}{10}\right)^2$$

$$R = 22\sqrt{\left(1 + \frac{\sqrt{3}}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = 25 \cdot 8 \text{ feet per second}$$

and its direction is at an angle θ below the horizontal, so that—

$$\tan \theta = \frac{Y}{X} = \frac{2 \cdot 2}{22 + 2 \cdot 2\sqrt{3}} = \frac{1}{10 + \sqrt{3}} = \frac{1}{11 \cdot 732} = 0 \cdot 0852$$

and $\theta = 4 \cdot 85°$

Examples II.

1. A point in the connecting rod of a steam engine moves forwards horizontally at 5 feet per second, and at the same time has a velocity of 3 feet per second in the same vertical plane, but in a direction inclined 110° to that of the horizontal motion. Find the magnitude and direction of its actual velocity.

2. A stone is projected at an angle of 36° to the horizontal with a velocity of 500 feet per second. Find its horizontal and vertical velocities.

3. In order to cross at right angles a straight river flowing uniformly at 2 miles per hour, in what direction should a swimmer head if he can get through still water at $2\frac{1}{2}$ miles per hour, and how long will it take him if the river is 100 yards wide?

4. A weather vane on a ship's mast points south-west when the ship is steaming due west at 16 miles per hour. If the velocity of the wind is 20 miles per hour, what is its true direction?

5. Two ships leave a port at the same time, the first steams north-west at 15 knots per hour, and the second 30° south of west at 17 knots. What is the speed of the second relative to the first? After what time will they be 100 knots apart, and in what direction will the second lie from the first?

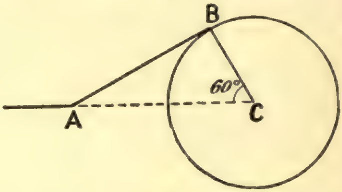
6. A ship steaming due east at 12 miles per hour crosses the track of another ship 20 miles away due south and going due north at 16 miles per hour. After what time will the two ships be a minimum distance apart, and how far will each have travelled in the interval.

7. Part of a machine is moving east at 10 feet per second, and after $\frac{1}{20}$ second it is moving south-east at 4 feet per second. What is the amount and direction of the average acceleration during the $\frac{1}{20}$ second?

8. How long will it take a body to slide down a smooth plane the length of which is 20 feet, the upper end being 3·7 feet higher than the lower one.

9. The minute-hand of a clock is 4 feet long, and the hour-hand is 3 feet long. Find in inches per minute the velocity of the end of the minute-finger relative to the end of the hour-hand at 3 o'clock and at 12 o'clock.

10. A crank, CB, is 1 foot long and makes 300 turns clockwise per minute. When CB is inclined 60° to the line CA, A is moving along AC



at a velocity of 32 feet per second. Find the velocity of the point B relative to A.

11. If a motor car is travelling at 30 miles per hour, and the wheels are 30 inches diameter, what is their angular velocity about their axes? If the car comes to rest in 100 yards under a uniform retardation, find the angular retardation of the wheels.

12. A flywheel is making 180 revolutions per minute, and after 20 seconds it is turning at 140 revolutions per minute. How many revolutions will it make, and what time will elapse before stopping, if the retardation is uniform?

CHAPTER II

THE LAWS OF MOTION

34. Newton's Laws of Motion were first put in their present form by Sir Isaac Newton, although known before his time. They form the foundation of the whole subject of dynamics.

35. First Law of Motion.—*Every body continues in its state of rest or uniform motion except in so far as it may be compelled by force to change that state.*

We know of no case of a body unacted upon by any force whatever, so that we have no direct experimental evidence of this law. In many cases the forces in a particular direction are small, and in such cases the change in that direction is small, *e.g.* a steel ball rolling on a horizontal steel plate. To such instances the second law is really applicable.

From the first law we may define *force* as that which tends to change the motion of bodies either in magnitude or direction.

36. Inertia.—It is a matter of everyday experience that some bodies take up a given motion more quickly than others under the same conditions. For example, a small ball of iron is more easily set in rapid motion by a given push along a horizontal surface than is a large heavy one. In such a case the larger ball is said to have more *inertia* than the small one. *Inertia* is, then, the property of resisting the taking up of motion.

37. Mass is the name given to inertia when expressed as a measurable quantity. The more matter there is in a body the greater its mass. The mass of a body depends upon its volume and its density being proportional to both. We may define density of a body as being its mass divided by its volume, or mass per unit volume in suitable units.

If m = the mass of a body,
v = its volume,
and ρ = its density,

$$\text{then } \rho = \frac{m}{v}$$

A common British unit of mass is one pound. This is often used in commerce, and also in one absolute system (British) of mechanical units; but we shall find it more convenient to use a unit about 32·2 times as large, for reasons to be stated shortly. This unit has no particular name in general use. It is sometimes called the gravitational unit of mass, or the "engineer's unit of mass."

In the c.g.s. (centimetre-gramme-second) absolute system, the unit mass is the gramme, which is about $\frac{1}{453 \cdot 6}$ lb.

38. The weight of a body is the force with which the earth attracts it. This is directly proportional to its mass, but is slightly different at different parts of the earth's surface.

39. Momentum is sometimes called the quantity of motion of a body. If we consider a body moving with a certain velocity, it has only half as much motion as two exactly similar bodies would have when moving at that velocity, so that the quantity of motion is proportional to the quantity of matter, *i.e.* to the mass. Again, if we consider the body moving with a certain velocity, it has only half the quantity of motion which it would have if its velocity were doubled, so that the quantity of motion is proportional also to the velocity. The quantity of motion of a body is then proportional to the product (mass) × (velocity), and this quantity is given the name *momentum*. The unit of momentum is, then, that possessed by a body of unit mass moving with unit velocity. It is evidently a vector quantity, since it is a product of velocity, which is a vector quantity, and mass, which is a scalar quantity, and its direction is that of the velocity factor. It can be resolved and compounded in the same way as can velocity.

40. Second Law of Motion.—*The rate of change of*

momentum is proportional to the force applied, and takes place in the direction of the straight line in which the force acts. This law states a simple relation between momentum and force, and, as we have seen how momentum is measured, we can proceed to the measurement of force.

The second law states that if F represents force—

$$F \propto \text{rate of change of } (m \times v)$$

where m = mass, v = velocity;

therefore $F \propto m \times$ (rate of change v), if m remains constant

$$\text{or } F \propto m \times f$$

where f = acceleration,

$$\text{and } f \propto \frac{F}{m}$$

where F is the resultant force acting on the mass m;

$$\text{hence } F = m \times f \times \text{a constant,}$$

and by a suitable choice of units we may make the constant unity, viz. by taking as unit force that which gives unit mass unit acceleration. We may then write—

$$\text{force} = (\text{mass}) \times (\text{acceleration})$$
$$\text{or } F = m \times f$$

If we take 1 lb. as unit mass, then the force which gives 1 lb. an acceleration of 1 foot per second per second is called the *poundal*. This system of units is sometimes called the *absolute system*.[1] This unit of force is not in general use with engineers and others concerned in the measurement and calculation of force and power, the general practice being to take the weight of 1 lb. at a fixed place as the unit of force. We call this a force of 1 lb., meaning a force equal to the weight of 1 lb. As mentioned in Art. 38, the weight of 1 lb. of matter varies slightly at different parts of the earth's surface, but the variation is not of great amount, and is usually negligible.

[1] The gravitational system is also really an absolute system, inasmuch as all derived units are connected to the fundamental ones by fixed physical relations. See Appendix.

41. Gravitational or Engineer's Units.—One pound of force acting on 1 lb. mass of matter (viz. its own weight) in London[1] gives it a vertical acceleration of about 32·2 feet per second per second, and since acceleration $= \frac{\text{force}}{\text{mass}}$, 1 lb. of force will give an acceleration of 1 foot per second per second (*i.e.* 32·2 times less), if it acts on a mass of 32·2 lbs. Hence, if we wish to have force defined by the relation—

$$\text{force} = \text{rate of change of momentum,}$$
$$\text{or force} = (\text{mass}) \times (\text{acceleration})$$
$$F = m \times f$$

we must adopt g lbs. as our unit of mass, where g is the acceleration of gravity in feet per second per second in some fixed place; the number 32·2 is correct enough for most practical purposes for any latitude. This unit, as previously stated, is sometimes called the engineers' unit of mass.

Then a body of weight w lbs. has a mass of $\frac{w}{g}$ units, and the equation of Art. 40 becomes $F = \frac{w}{g} \times f$.

Another plan is to merely adopt the relation, force = (mass) × (acceleration) × constant. The mass is then taken in pounds. and if the force is to be in pounds weight (and not in *poundals*) the constant used is g (32·2). There is a strong liability to forget to insert the constant g in writing expressions for quantities involving force, so we shall adopt the former plan of using 32·2 lbs. as the unit of mass. The unit of momentum is, then, that possessed by 32·2 lbs. moving with a velocity of 1 foot per second, and the unit force the weight of 1 lb. The number 32·2 will need slight adjustment for places other than London, if very great accuracy should be required.

Defining unit force as the weight of 1 lb. of matter, we may define the gravitational unit of mass as that mass which has unit acceleration under unit force.

42. C.G.S. (centimetre-gramme-second) Units.—In this absolute system the unit of mass is the gramme; the

[1] The place chosen is sometimes quoted as sea-level at latitude 45°.

unit of momentum that in 1 gramme moving at 1 centimetre per second; and the unit of force called the *dyne* is that necessary to accelerate 1 gramme by 1 centimetre per second per second. The weight of 1 gramme is a force of about 981 dynes, since the acceleration of gravity is about 981 centimetres per second per second (981 centimetres being equal to about 32·2 feet).

The weight of one kilogram (1000 grammes) is often used by Continental engineers as a unit of force.

Example 1.—A man pushes a truck weighing 2·5 tons with a force of 40 lbs., and the resistance of the track is equivalent to a constant force of 10 lbs. How long will it take to attain a velocity of 10 miles per hour? The constant effective forward force is $40 - 10 = 30$ lbs., hence the acceleration is—

$$\frac{\text{force}}{\text{mass}} = 30 \div \frac{2·5 \times 2240}{32·2} = 0·1725 \text{ foot per second per second}$$

$$10 \text{ miles per hour} = \tfrac{88}{6} \text{ or } \tfrac{44}{3} \text{ feet per second}$$

The time to generate this velocity at 0·1725 foot per second per second is then $\frac{44}{3} \div 0·1725 = 85$ seconds, or 1 minute 25 seconds.

Example 2.—A steam-engine piston, weighing 75 lbs., is at rest, and after 0·25 second it has attained a velocity of 10 feet per second. What is the average accelerating force acting on it during the 0·25 second?

$$\text{Average acceleration} = 10 \div 0·25 = 40 \text{ feet per sec. per sec.}$$

hence average accelerating force is $\frac{75}{32·2} \times 40 = 93·2$ lbs.

43. We have seen that by a suitable choice of units the force acting on a body is numerically equal to its rate of change of momentum; the second law further states that the force and the change of momentum are in the same direction. Momentum is a vector quantity, and therefore change of momentum must be estimated as a vector change having magnitude and direction.

For example, if the momentum of a body is represented by *ab* (Fig. 25), and after *t* seconds it is represented by *cd*, then the change of momentum in *t* seconds is $cd - ab = eg$ (see

Art. 20), where $ef = cd$ and $gf = ab$. Then the average rate of change of momentum in t seconds is represented by $\frac{eg}{t}$ in magnitude and direction, *i.e.* the resultant force acting on the body during the t seconds was in the direction eg. Or Fig. 25

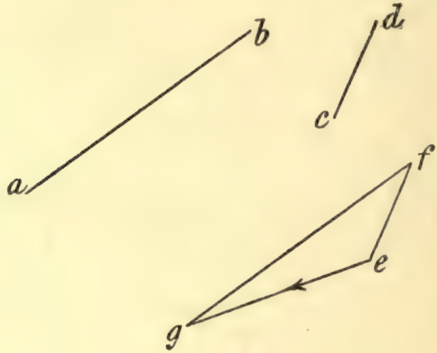


FIG. 25.

may be taken as a vector diagram of velocities, and eg as representing change of velocity. Then $\frac{eg}{t}$ represents acceleration, and multiplied by the mass of the body it represents the average force.

Example.—A piece of a machine weighing 20 lbs. is at a certain instant moving due east at 10 feet per second, and after 1·25 seconds it is moving south-east at 5 feet per second. What was the average force acting on it in the interval?

The change of momentum per second may be found directly, or the change of velocity per second may be found, which, when multiplied by the (constant) mass, will give the force acting.

Using the method of resolution of velocities, the

final component of velocity E. $= 5 \cos 45^\circ = \frac{5}{\sqrt{2}}$ feet per second

initial ,, ,, E. $= 10$,, ,,

hence gain of component velocity $\Big\} = \left(\frac{5}{\sqrt{2}} - 10\right)$ east, or $\left(10 - \frac{5}{\sqrt{2}}\right)$ west

Again, the gain of velocity south is $5 \sin 45^\circ = \frac{5}{\sqrt{2}}$ feet per second

If R = resultant change of velocity—

$$R^2 = \left(\frac{5}{\sqrt{2}}\right)^2 + \left(10 - \frac{5}{\sqrt{2}}\right)^2 = 54{\cdot}3$$

and $R = \sqrt{54{\cdot}3} = 7{\cdot}37$ feet per second in $1\frac{1}{4}$ seconds

Hence acceleration $= 7{\cdot}37 \div 1{\cdot}25 = 5{\cdot}9$ feet per second per second, and average force acting $= \frac{20}{32{\cdot}2} \times 5{\cdot}9 = 3{\cdot}66$ lbs. in a direction south of west at an angle whose tangent is $\frac{5}{\sqrt{2}} \div \left(10 - \frac{5}{\sqrt{2}}\right)$ or $0{\cdot}546$, which is an angle of about $28\frac{1}{2}°$ south of west (by table of tangents).

44. Triangle, Polygon, etc., of Forces.—It has been seen (Art. 27) that acceleration is a vector quantity having magnitude and direction, and that acceleration can be compounded and resolved by means of vectors. Also (Art. 40) that force is the product of acceleration and mass, the latter being a mere magnitude or scalar quantity; hence force is a vector quantity, and concurrent forces can be compounded by vector triangles or polygons such as were used in Arts. 19 and 24, and resolved into components as in Arts. 25 and 28.

We are mainly concerned with uniplanar forces, but the methods of resolution, etc., are equally applicable to forces in different planes; the graphical treatment would, however, involve the application of solid geometry.

The particular case of bodies subject to the action of several forces having a resultant zero constitutes the subject of Statics.

The second law of motion is true when the resultant force is considered or when the components are considered, *i.e.* the rate of change of momentum in any particular direction is proportional to the component force in that direction.

45. Impulse.—By the impulse of a constant force in any interval of time, we mean the product of the force and time. Thus, if a constant force of F pounds act for t seconds, the impulse of that force is $F \times t$. If this force F has during the interval t acted without resistance on a mass m, causing its velocity to be accelerated from v_1 to v_2, the change of momentum

during that time will have been from mv_1 to mv_2, *i.e.* $mv_2 - mv_1$ or $m(v_2 - v_1)$. And the change of velocity in the interval t under the constant acceleration f is $f \times t$ (Art. 11), therefore $v_2 - v_1 = ft$, and $m(v_2 - v_1) = m . f . t$; but $m \times f = \text{F}$, the accelerating force (by Art. 40), hence $m(v_2 - v_1) = \text{F}t$, or, in words, the change of momentum is equal to the impulse. The force, impulse, and change of momentum are all to be estimated in the same direction.

The impulse may be represented graphically as in Fig. 26. If ON represents t seconds, and PN represents F lbs. to scale,

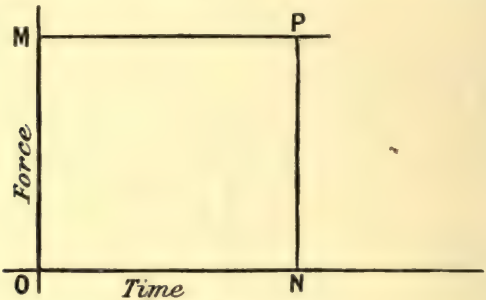


FIG. 26.

then the area MPNO under the curve MP of constant force represents $\text{F} \times t$, the impulse, and therefore also the change of momentum.

Impulse of a Variable Force.—In the case of a variable force the interval of time is divided into a number of parts, and the impulse calculated during each as if the force were constant during each of the smaller intervals, and equal to some value which it actually has in the interval. The sum of these impulses is approximately the total impulse during the whole time. We can make the approximation as near as we please by taking a sufficiently large number of very small intervals. The graphical representation will illustrate this point.

Fig. 27 shows the varying force F at all times during the interval NM. Suppose the interval NM divided up into a

number of small parts such as CD. Then AC represents the force at the time OC; the force is increasing, and therefore in the interval CD the impulse will be greater than that represented by the rectangle AEDC, and less than that represented by the rectangle FBDC. The total impulse during the interval NM is similarly greater than that represented by a series of rectangles such as AEDC, and less than that represented by a series of rectangles such as FBDC. Now, if we consider the number of rectangles to be indefinitely increased, and the width of each rectangle to be decreased indefinitely, the area PQMN under the curve PQ is the area which lies always between the sums of the areas of the two series of rectangles however far the subdivision may be carried, and therefore it represents the total impulse in the time NM, and therefore also the gain of momentum in that time.

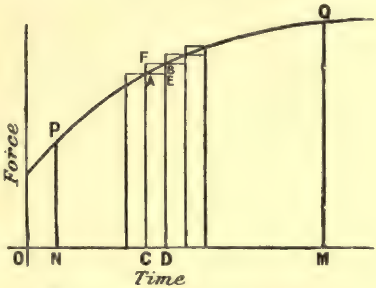


FIG. 27.—Impulse of a variable force.

It may be noticed that the above statement agrees exactly with that made in Art. 16. In Fig. 8 the vertical ordinates are similar to those in Fig. 27 divided by the mass, and the gain of velocity represented by the area under PQ in Fig. 8 is also similar to the gain of momentum divided by the mass.

Note that the force represented by $\dfrac{\text{area PQMN}}{\text{length NM}}$ (*i.e.* by the average height of the PQMN) is the mean force or time-average of the force acting during the interval NM. This time-average force may be defined as $\dfrac{\text{total impulse}}{\text{total time}}$.

The area representing the impulse of a negative or opposing force will lie below the line OM in a diagram such as Fig. 27. In case of a body such as part of a machine starting from rest and coming to rest again, the total change of momentum is zero; then as much area of the force-time diagram lies below

the time base line (OM) as above it. The reader should sketch out such a case, and the velocity-time or momentum-time curve to be derived from it, by the method of Art. 16, and carefully consider the meaning of all parts of the diagrams—the slopes, areas, changes of sign, etc.

The slope of a momentum-time curve represents accelerating force just as that of a velocity-time represents acceleration (see Art. 14), the only difference in the case of momentum and force being that mass is a factor of each.

It is to be noticed that the impulse or change of momentum in a given interval is a vector quantity having definite direction. It must be borne in mind that the change of momentum is in the same direction as the force and impulse. If the force varies in direction it may be split into components (Art. 44), and the change of momentum in two standard directions may be found, and the resultant of these would give the change of momentum in magnitude and direction.

46. Impulsive Forces.—Forces which act for a very short time and yet produce considerable change of momentum on the bodies on which they act are called *impulsive* forces. The forces are large and the time is small. Instances occur in blows and collisions.

47. The second law of motion has been stated, in Art. 40, in terms of the *rate* of change of momentum. It can now be stated in another form, viz. *The change of momentum is equal to the impulse of the applied force, and is in the same direction.*

Or in symbols, for a mass m—

$$m(v_2 - v_1) = \mathrm{F} \, . \, t$$

where v_2 and v_1 are the final and initial velocities, the subtraction being performed geometrically (Art. 20), and F is the *mean force* acting during the interval of time t.

Example 1.—A body weighing W lbs. is set in motion by a uniform net force P_1 lbs., and in t_1 seconds it attains a velocity V feet per second. It then comes to rest in a further period of t_2 seconds under the action of a uniform retarding force of P_2 lbs. Find the relation between P_1, P_2, and V.

During the acceleration period the gain of momentum in the direction of motion is $\frac{W}{g} \cdot V$ units, and the impulse in that direction is $P_1 t_1$, hence—

$$P_1 t_1 = \frac{W}{g} \cdot V$$

During retardation the *gain* of momentum *in the direction of motion* is $-\frac{W}{g} \cdot V$ units, and the impulse *in that direction* is $-P_2 \cdot t_2$; hence—

$$P_2 t_2 = \frac{W}{g} \cdot V$$

$$\text{and finally } \frac{W}{g} \cdot V = P_1 t_1 = P_2 t_2 = \frac{P_1 P_2}{P_1 + P_2}(t_1 + t_2)$$

the last relation following algebraically from the two preceding ones.

Example 2.—If a locomotive exerts a constant draw-bar pull of 4 tons on a train weighing 200 tons up an incline of 1 in 120, and the resistance of the rails, etc., amounts to 10 lbs. per ton, how long will it take to attain a velocity of 25 miles per hour from rest, and how far will it have moved?

The forces resisting acceleration are—

		lbs.
(*a*) Gravity $\frac{1}{120}$ of 200 tons (see Art. 28)	$= \frac{200 \times 2240}{120}$	$= 3733$
(*b*) Resistance at 10 lbs. per ton, 200×10		$= 2000$
Total		5733

The draw-bar pull is $4 \times 2240 = 8960$ lbs.; hence the net accelerating force is $8960 - 5733 = 3227$ lbs.

Let t be the required time in seconds; then the impulse is $3227 \times t$ units.

25 miles per hour $= \frac{5}{12} \times 88$ feet per second (88 feet per second $= 60$ miles per hour)

so that the gain of momentum is $\frac{W}{g} \cdot V$—

$$\frac{200 \times 2240}{32 \cdot 2} \times \frac{5}{12} \times 88$$

therefore—

$$3227 \cdot t = \frac{200 \times 2240}{32 \cdot 2} \times \frac{5}{12} \times 88$$

from which $t = 159$ seconds, or 2 minutes 39 seconds

Since the acceleration has been uniform, the average speed is half the maximum (Art. 28), and the distance travelled will be in feet—

$$\tfrac{1}{2} \times \tfrac{5}{12} \times 88 \times 159 = 2915 \text{ feet}$$

Example 3.—How long would it take the train in Ex. 2 to go 1 mile up the incline, starting from rest and coming to rest at the end without the use of brakes?

Let t_1 = time occupied in acceleration,

t_2 = time occupied in retardation.

During the retardation period the retarding force will be as in Ex. 2, a total of 5733 lbs. after acceleration ceases. The average velocity during both periods, and therefore during the whole time, will be half the maximum velocity attained.

$$\text{Average velocity} = \frac{5280}{t_1 + t_2} \text{ feet per second}$$

$$\text{and maximum velocity} = 2 \times \frac{5280}{t_1 + t_2} \text{ feet per second}$$

$$\therefore \text{momentum generated} = \frac{200 \times 2240}{32 \cdot 2} \times 2 \times \frac{5280}{t_1 + t_2} \text{ units}$$

$$\text{The impulse} = 3227t_1 = 5733t_2$$

$$\therefore t_1 = \frac{5733}{3227}t_2$$

$$\text{and } t_1 + t_2 = \left(\frac{5733}{3227} + 1\right)t_2 = \frac{8960}{3227}t_2$$

$$\text{and } t_2 = \frac{3227}{8960}(t_1 + t_2)$$

By the second law, change of momentum = impulse.

$$\therefore \frac{200 \times 2240}{32 \cdot 2} \times 2 \times \frac{5280}{(t_1 + t_2)} = 5733t_2$$

and substituting for t_2 the value found—

$$\frac{200 \times 2240}{32 \cdot 2} \times 2 \times \frac{5280}{t_1 + t_2} = 5733 \times \frac{3227}{8960}(t_1 + t_2)$$

agreeing with the last result in Ex. 1.

$$\text{hence } (t_1 + t_2) = 267 \text{ seconds} = 4 \text{ minutes } 27 \text{ seconds}$$

Example 4.—A car weighing 12 tons starts from rest, and has a constant resistance of 500 lbs. The tractive force, F, on the car after t seconds is as follows:—

t ...	0	2	5	8	11	13	16	19	20
F ...	1280	1270	1220	1110	905	800	720	670	660

Find the velocity of the car after 20 seconds from rest, and show how to find the velocity at any time after starting, and to find the distance covered up to any time.

Plot the curve of F and t, as in Fig. 28, and read off the force,

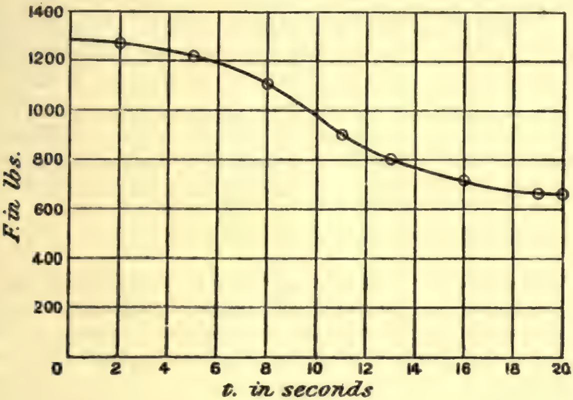


FIG. 28.

say every 4 seconds, starting from $t = 2$, and subtract the 500 lbs. resistance from each as follows:—

t	2	6	10	14	18
F lbs. ...	1270	1190	980	760	680
F − 500 ...	770	690	480	260	180

The mean accelerating force in the first 4 seconds is approximately 770 lbs., and therefore the impulse is 770 × 4, which is also the gain of momentum.

The mass of the car is $\frac{12 \times 2240}{32 \cdot 2} = 835$ units

$$\text{The velocity after 4 seconds} = \frac{\text{momentum}}{\text{mass}} = \frac{770 \times 4}{835}$$
$$= 3{\cdot}69 \text{ feet per second}$$

Similarly, finding the momentum and gain of velocity in each 4 seconds, we have—

t	0	4	8	12	16	20
Gain of momentum in 4 seconds ...	0	3080	2760	1920	1040	720
Momentum	0	3080	5840	7760	8800	9520
Velocity, feet per second	0	3·69	7·00	9·31	10·55	11·41

After 20 seconds the velocity is approximately 11·41 feet per second. The velocity after any time may be obtained approximately by plotting a curve of velocities and times from the values obtained, and reading intermediate values. More points on the velocity-time curves may be found if greater accuracy be desired.

The space described is represented by the area under the velocity-time curve, and may be found as in Art. 14.

EXAMPLES III.

1. The moving parts of a forging hammer weigh 2 tons, and are lifted vertically by steam pressure and then allowed to fall freely. What is the momentum of the hammer after falling 6 feet? If the force of the blow is expended in 0·015 second, what is the average force of the blow?

2. A mass of 50 lbs. acquires a velocity of 25 feet per second in 10 seconds, and another of 20 lbs. acquires a velocity of 32 feet per second in 6 seconds. Compare the forces acting on the two masses.

3. A constant unresisted force of 7000 dynes acts on a mass of 20 kilograms for 8 seconds. Find the velocity attained in this time.

4. A train weighing 200 tons has a resistance of 15 lbs. per ton, supposed constant at any speed. What tractive force will be required to give it a velocity of 30 miles per hour in 1·5 minutes?

5. A jet of water of circular cross-section and 1·5 inches diameter impinges on a flat plate at a velocity of 20 feet per second, and flows off at right angles to its previous path. How much water reaches the plate per second? What change of momentum takes place per second, and what force does the jet exert on the plate?

6. A train travelling at 40 miles per hour is brought to rest by a uniform resisting force in half a mile. How much is the total resisting force in pounds per ton?

7. A bullet weighing 1 oz. enters a block of wood with a velocity of 1800 feet per second, and penetrates it to a depth of 8 inches. What is the average resistance of the wood in pounds to the penetration of the bullet?

8. The horizontal thrust on a steam-engine crank-shaft bearing is 10 tons,

and the dead weight it supports vertically is 3 tons. Find the magnitude and direction of the resultant force on the bearing.

9. A bullet weighing 1 oz. leaves the barrel of a gun 3 feet long with a velocity of 1500 feet per second. What was the impulse of the force produced by the discharge? If the bullet took 0·004 second to traverse the barrel, what was the average force exerted on it?

10. A car weighing 10 tons starts from rest. During the first 25 seconds the average drawing force on the car is 750 lbs., and the average resistance is 40 lbs. per ton. What is the total impulse of the effective force at the end of 25 seconds, and what is the speed of the car in miles per hour?

11. The reciprocating parts of a steam-engine weigh 483 lbs., and during one stroke, which occupies 0·3 second, the velocities of these parts are as follows:—

Time	0·0	0·025	0·05	0·075	0·100	0·125	0·150	0·175	0·200	0·225	0·250	0·275	0·300
Velocity in feet per sec.	0·00	3·46	6·55	8·91	10·22	10·90	10·48	9·32	7·75	6·02	4·14	2·10	0·00

Find the force necessary to give the reciprocating parts this motion, and draw a curve showing its values on a time base throughout the stroke. Draw a second curve showing the distances described from rest, for every instant during the stroke. From these two curves a third may be drawn, showing the accelerating force on the reciprocating parts, on the distance traversed as a base.

48. Third Law of Motion.—*To every action there is an equal and opposite reaction.* By the word "action" here is meant the exertion of a force. We may state this in another way. If a body A exerts a certain force on a body B, then B exerts on A a force of exactly equal magnitude, but in the opposite direction.

The medium which transmits the equal and opposite forces is said to be in a state of *stress*. (It will also be in a state of *strain*, but this term is limited to deformation which matter undergoes under the influence of stress.)

Suppose A and B (Fig. 29) are connected by some means (such as a string) suitable to withstand tension, and A exerts a pull T on B. Then B exerts an equal tension T′ on A. This will be true whether A moves B or not. Thus A may be a locomotive, and B a train, or A may be a ship moored to

a fixed post, B. Whether A moves B or not depends upon what other forces may be acting on B.

Again, if the connection between A and B can transmit a

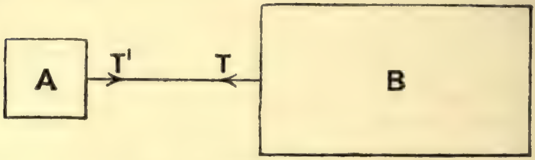


FIG. 29.—Connection in tension.

thrust (Fig. 30), A may exert a push P on B. Then B exerts an equal push P′ on A. As an example, A may be a gun, and B a projectile; the gases between them are in compression.

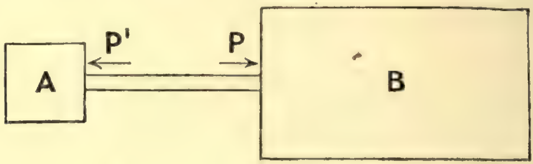


FIG. 30.—Connection in compression.

Or in a case where motion does not take place, A may be a block of stone resting on the ground B; then A and B are in compression at the place of contact.

49. An important consequence of the third law is that the total momentum of the two bodies is unaltered by any mutual action between them. For since the force exerted by A on B is the same as that exerted by B on A, the impulse during any interval given by A to B is of the same amount as that given by B to A and in the opposite direction. Hence, if B gains any momentum A loses exactly the same amount, and the total change of momentum is zero, and this is true for any and every direction. This is expressed by the statement that for any isolated system of bodies momentum is conservative. Thus when a projectile is fired from a cannon, the impulse or change of momentum of the shot due to the explosion is of equal amount to that of the recoiling cannon in the opposite

direction. The momentum of the recoil is transmitted to the earth, and so is that of the shot, the net momentum given to the earth being also zero.

50. Motion of Two Connected Weights.—Suppose two weights, W_1 lbs. and W_2 lbs., to be connected by a light inextensible string passing over a small and perfectly smooth pulley, as in Fig. 31. If W_1 is greater than W_2, with what acceleration (f) will they move (W_1 downwards and W_2 upwards), and what will be the tension (T) of the string?

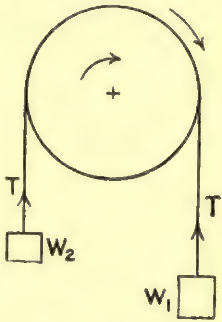


FIG. 31.

Consider W_1 $\left(\text{of mass } \frac{W_1}{g}\right)$: the downward force on it is W_1 (its weight), and the upward force is T, which is the same throughout the string by the "third law;" hence the downward accelerating force is $W_1 - T$.

Hence (by Art. 40) $\frac{W_1}{g}.f = W_1 - T$ . . . . . . (1)

Similarly, on W_2 the *upward* accelerating force is $T - W_2$;

$$\text{hence } \frac{W_2}{g}.f = T - W_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

adding (1) and (2)—

$$\frac{W_1 + W_2}{g}.f = W_1 - W_2$$

$$\text{or } f = \frac{W_1 - W_2}{W_1 + W_2} \cdot g$$

and from (1)—

$$T = W_1\left(1 - \frac{f}{g}\right) = \frac{2W_1W_2}{W_1 + W_2}$$

The acceleration f might have been stated from considering the two weights and string as one complete system. The accelerating force on which is $W_1 - W_2$, and the mass of which is $\frac{W_1 + W_2}{g}$;

$$\text{hence } f = \frac{\text{accelerating force}}{\text{total mass}} = \frac{W_1 - W_2}{W_1 + W_2} \cdot g$$

As a further example, suppose W_2 instead of being suspended slides along a perfectly smooth horizontal table as in Fig. 32,

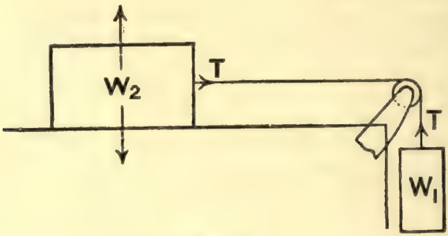


FIG. 32.

the accelerating force is W_1, and the mass in motion is $\frac{W_1 + W_2}{g}$

hence the acceleration $f = \frac{W_1}{W_1 + W_2} \cdot g$

$$\text{and since } f \text{ also} = \frac{\text{accelerating force on } W_2}{\text{mass of } W_2} = \frac{T}{W_2} \cdot g$$

$$\text{we have } T = \frac{W_1 W_2}{W_1 + W_2}$$

If the motion of W_2 were opposed by a horizontal force, F, the acceleration would be $\frac{W_1 - F}{W_1 + W_2} \cdot g$.

We have left out of account the weight of W_2 and the reaction of the table. These are equal and opposite, and neutralize each other. The reaction of the pulley on the string is normal to the direction of motion, and has therefore no accelerating effect.

Atwood's Machine is an apparatus for illustrating the laws of motion under gravity. It consists essentially of a light, free pulley and two suspended weights (Fig. 31), which can be made to differ by known amounts, a scale of lengths, and clockwork to measure time. Quantitative measurements of acceleration of known masses under the action of known accelerating forces can be made. Various corrections are

necessary, and this method is not the one adopted for measuring the acceleration g.

Example 1.—A hammer weighing W lbs. strikes a nail weighing w lbs. with a velocity V feet per second and does not rebound. The nail is driven into a fixed block of wood which offers a uniform resistance of P lbs. to the penetration of the nail. How far will the nail penetrate the fixed block?

Let V' = initial velocity of nail after blow.

$$\text{Momentum of hammer before impact} = \frac{W}{g} . V$$

$$\text{momentum of hammer and nail after impact} = \frac{W + w}{g} . V'$$

$$\text{hence } \frac{W + w}{g} . V' = \frac{W}{g} V \qquad \therefore V' = \frac{W}{w + W} . V$$

Let t = time of penetration.

$$\text{Impulse } Pt = \frac{W}{g} . V \text{ (the momentum overcome by P)}$$

$$\therefore t = \frac{WV}{gP}$$

During the penetration, average velocity $= \frac{1}{2}V'$ (Arts. 11 and 14)

hence distance moved by nail $= \frac{1}{2}V' \times t$

$$= \frac{1}{2} . \frac{W}{W + w} V \times \frac{WV}{gP}$$

$$= \frac{1}{2}\frac{V^2}{gP} . \frac{W^2}{W + w}$$

Example 2.—A cannon weighing 30 tons fires a 1000-lb. projectile with a velocity of 1000 feet per second. With what initial velocity will the cannon recoil? If the recoil is overcome by a (time) average force of 60 tons, how far will the cannon travel? How long will it take?

Let V = initial velocity of cannon in feet per second.

$$\text{Momentum of projectile} = \frac{1000}{g} \times 1000 = \text{momentum of cannon}$$

$$\text{or } \frac{1000}{g} \times 1000 = \frac{30 \times 2240}{g} \times V$$

$$\text{and } V = \frac{1000 \times 1000}{30 \times 2240} = 14{\cdot}87 \text{ feet per second}$$

Let t = time of recoil.

Impulse of retarding force = $60 \times 2240 \times t$ = momentum of shot

$$60 \times 2240 \times t = \frac{1000 \times 1000}{32 \cdot 2}$$

and hence $t = 0 \cdot 231$ second

$$\text{Distance moved} = \tfrac{1}{2}V \times t = \frac{14 \cdot 87 \times 0 \cdot 231}{2} = 1 \cdot 74 \text{ feet}$$

Example 3.—Two weights are connected by a string passing over a light frictionless pulley. One is 12 lbs. and the other 11 lbs. They are released from rest, and after 2 seconds 2 lbs. are removed from the heavier weight. How soon will they be at rest again, and how far will they have moved between the instant of release and that of coming to rest again?

First period.

$$\text{Acceleration} = \frac{\text{accelerating force}}{\text{total mass}} = \frac{12 - 11}{12 + 11} \times g = \frac{g}{23}$$

$$\text{velocity after 2 seconds} = 2 \times \frac{32 \cdot 2}{23} = 2 \cdot 8 \text{ feet per second}$$

Second period.

$$\text{Retardation} = \frac{11 - 10}{11 + 10} \times g = \frac{g}{21}$$

$$\text{time to come to rest} = \frac{\text{velocity}}{\text{retardation}} = \frac{2 \times \frac{g}{23}}{\frac{g}{21}} = 2 \times \tfrac{21}{23} = 1 \cdot 825 \text{ sec.}$$

average velocity throughout = $\frac{1}{2}$ maximum velocity (Art. 11)

total time = $2 + 1 \cdot 825$ seconds

distance moved = $\frac{1}{2} \times 2 \cdot 8 \times 3 \cdot 825 = 5 \cdot 35$ feet

Examples IV.

1. A fireman holds a hose from which a jet of water 1 inch in diameter issues at a velocity of 80 feet per second. What thrust will the fireman have to exert in order to support the jet?

2. A machine-gun fires 300 bullets per minute, each bullet weighing 1 oz. If the bullets have a horizontal velocity of 1800 feet per second, find the average force exerted on the gun.

3. A pile-driver weighing W lbs. falls through h feet and drives a pile weighing w lbs. a feet into the ground. Show that the average force of the blow is $\frac{W^2}{W + w} \cdot \frac{h}{a}$ lbs.

4. A weight of 5 cwt. falling freely, drives a pile weighing 600 lbs. 2 inches into the earth against an average resistance of 25 tons. How far will the weight have to fall in order to do this?

5. A cannon weighing 40 tons projects a shot weighing 1500 lbs. with a velocity of 1400 feet per second. With what initial velocity will the cannon recoil? What average force will be required to bring it to rest in 3 feet?

6. A cannon weighing 40 tons has its velocity of recoil destroyed in 2 feet 9 inches by an average force of 70 tons. If the shot weighed 14 cwt., find its initial velocity.

7. A lift has an upward acceleration of 3·22 feet per second per second. What pressure will a man weighing 140 lbs. exert on the floor of the lift? What pressure would he exert if the lift had an acceleration of 3·22 feet per second per second downward? What upward acceleration would cause his weight to exert a pressure of 170 lbs. on the floor?

8. A pit cage weighs 10 cwt., and on approaching the bottom of the shaft it is brought to rest, the retardation being at the rate of 4 feet per second per second. Find the tension in the cable by which the cage is lowered.

9. Two weights, one of 16 lbs. and the other of 14 lbs., hanging vertically, are connected by a light inextensible string passing over a perfectly smooth fixed pulley. If they are released from rest, find how far they will move in 3 seconds. What is the tension of the string?

10. A weight of 17 grammes and another of 20 grammes are connected by a fine thread passing over a light frictionless pulley in a vertical plane. Find what weight must be added to the smaller load 2 seconds after they are released from rest in order to bring them to rest again in 4 seconds. How many centimetres will the weights have moved altogether?

11. A weight of 5 lbs. hangs vertically, and by means of a cord passing over a pulley it pulls a block of iron weighing 10 lbs. horizontally along a table-top against a horizontal resistance of 2 lbs. Find the acceleration of the block and tension of the string.

12. What weight hanging vertically, as in the previous question, would give the 10-lb. block an acceleration of 3 feet per second per second on a perfectly smooth horizontal table?

13. A block of wood weighing 50 lbs. is on a plane inclined 40° to the horizontal, and its upward motion along the plane is opposed by a force of 10 lbs. parallel to the plane. A cord attached to the block, running parallel to the plane and over a pulley, carries a weight hanging vertically. What must this weight be if it is to haul the block 10 feet upwards along the plane in 3 seconds from rest?

CHAPTER III

WORK, POWER, AND ENERGY

51. Work.—When a force acts upon a body and causes motion, it is said to do work.

In the case of constant forces, work is measured by the product of the force and the displacement, one being estimated by its component in the direction of the other.

One of the commonest examples of a force doing work is that of a body being lifted against the force of gravity, its weight. The work is then measured by the product of the weight of the body, and the vertical height through which it is lifted. If we draw a diagram (Fig. 33) setting off the constant force F by a vertical ordinate, OM, then the work done during any displacement represented by ON is proportional to the area MPNO, and is represented by that area. If the scale of force is 1 inch $= p$ lbs., and the scale of distance is 1 inch $= q$ feet, then the scale of work is 1 square inch $= pq$ foot-lbs.

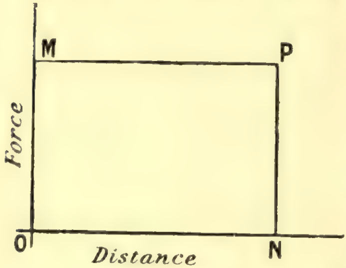


FIG. 33.—Work of a constant force.

52. Units of Work.—Work being measured by the product of force and length, the unit of work is taken as that done by a unit force acting through unit distance. In the British gravitational or engineer's system of units, this is the work done by a force of 1 lb. acting through a distance of 1 foot. It is called the foot-pound of work. If a weight

W lbs. be raised vertically through h feet, the work done is Wh foot-lbs.

Occasionally inch-pound units of work are employed, particularly when the displacements are small.

In the C.G.S. system the unit of work is the *erg*. This is the work done by a force of one dyne during a displacement of 1 centimetre in its own direction (see Art. 42).

53. Work of a Variable Force.—If the force during any displacement varies, we may find the total work done approximately by splitting the displacement into a number of parts and finding the work done during each part, as if the force during the partial displacement were constant and equal to some value it has during that part, and taking the sum of all the work so calculated in the partial displacements. We can make the approximation as near as we please by taking a sufficiently large number of parts. We may define the work actually done by the variable force as the limit to which such a sum tends when the subdivisions of the displacement are made indefinitely small.

54. Graphical Representation of Work of a Variable Force.—Fig. 34 is a diagram showing by its vertical ordinates

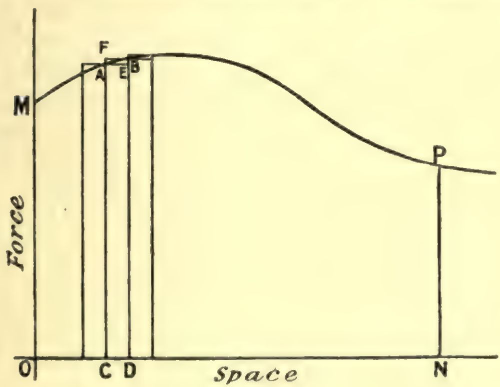


FIG. 34.

the force acting on a body, and by its horizontal ones the displacements. Thus, when the displacement is represented by

ON, the force acting on the body is represented by PN. Suppose the interval ON divided up into a number of small parts, such as CD. The force acting on the body is represented by AC when the displacement is that represented by OC. Since the force is increasing with increase of displacement the work done during the displacement CD is greater than that represented by the rectangle AEDC, and less than that represented by the rectangle FBDC. The total work done during the displacement will lie between that represented by the series of smaller rectangles, such as AEDC, and that represented by the series of larger rectangles, such as FBDC. The area MPNO under the curve MP will always lie between these total areas, and if we consider the number of subdivisions of ON to be carried higher indefinitely, the same remains true both of the total work done and the area under the curve MP. Hence the area MPNO under the curve MP represents the work done by the force during the displacement represented by ON.

The Indicator Diagram, first introduced by Watt for use on the steam-engine, is a diagram of the same kind as Fig. 34. The vertical ordinates are proportional to the total force exerted by the steam on the piston, and the horizontal ones are proportional to the displacement of the piston. The area of the figure is then proportional to the work done by the steam on the piston.

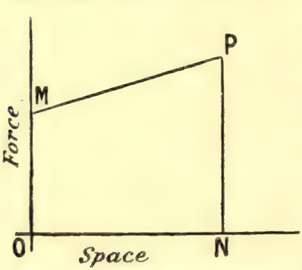


FIG. 35.—Force varying uniformly with space.

In the case of a force varying uniformly with the displacement, the curve MP is a straight line (Fig. 35), and the area $\text{MPNO} = \dfrac{\text{OM} + \text{PN}}{2} \times \text{ON}$, or if the initial force (OM) is F_1 lbs., and the final one (PN) is F_2 lbs., and the displacement (ON) is d feet, the work done is $\dfrac{F_1 + F_2}{2} . d$ foot-lbs.

In stretching an unstrained elastic body, such as a spring,

the force starts from zero (or $F_1 = 0$). Then the total work done is $\frac{1}{2}F_2d$, where F_2 is the greatest force exerted, and d is the amount of stretch.

Average Force.—The whole area MPNO (Figs. 34 and 35) divided by the above ON gives the mean height of the area; this represents the *space-average* of the force during the displacement ON. This will not necessarily be the same as the *time-average* (Art. 45). We may define the space-average of a varying force as the work done divided by the displacement.

55. Power.—Power is the rate of doing work, or the work done per unit of time.

One foot-pound per second might be chosen as the unit of power. In practice a unit 550 times larger is used; it is called the *horse-power*. It is equal to a rate of 550 foot-lbs. per second, or 33,000 foot-lbs. per minute. In the C.G.S. system the unit of power is not usually taken as one erg per second, but a multiple of this small unit. This larger unit is called a watt, and it is equal to a rate of 10^7 ergs per second. Engineers frequently use a larger unit, the kilowatt, which is 1000 watts. One horse-power is equal to 746 watts or 0·746 kilowatt.

Example 1.—A train ascends a slope of 1 in 85 at a speed of 20 miles per hour. The total weight of the train is 200 tons, and resistance of the rails, etc., amounts to 12 lbs. per ton. Find the horse-power of the engine.

The total force required to draw the load is—

$$(200 \times 12) + \frac{200 \times 2240}{85} = 7670 \text{ lbs.}$$

The number of feet moved through per minute is $\frac{1}{3} \times 88 \times 60 = 1760$ feet; hence the work done per minute is $1760 \times 7670 = 13,500,000$ foot-lbs., and since 1 horse-power = 33,000 foot-lbs. per minute, the H.P. is $\frac{13500000}{33000} = 409$ horse-power.

Example 2.—A motor-car weighing 15 cwt. just runs freely at 12 miles per hour down a slope of 1 in 30, the resistance at this speed just being sufficient to prevent any acceleration. What horse-power will it have to exert to run up the same slope at the same speed?

In running down the slope the propelling force is that of gravity, which is $\frac{1}{30}$ of the weight of the car (Arts. 28 and 44); hence the

resistance of the road is also (at 12 miles per hour) equivalent to $\frac{15 \times 112}{30}$ or 56 lbs.

Up the slope the opposing force to be overcome is 56 lbs. road resistance and 56 lbs. gravity (parallel to the road), and the total 112 lbs.

The distance travelled per minute at 12 miles per hour is $\frac{1}{5}$ mile = $\frac{5280}{5}$ or 1056 feet; hence the work done per minute is 112×1056 foot-lbs., and the H.P. is $\frac{112 \times 1056}{33000}$ or 3·584 H.P.

Example 3.—The spring of a safety-valve is compressed from its natural length of 20 inches to a length of 17 inches. It then exerts a force of 960 lbs. How much work will have to be done to compress it another inch, *i.e.* to a length of 16 inches?

The force being proportional to the displacement, and being 960 lbs. for 3 inches, it is $\frac{960}{3}$ or 320 lbs. per inch of compression.

When 16 inches long the compression is 4 inches, hence the force is 4×320 or 1280 lbs.; hence the work done in compression is $\frac{960 + 1280}{2} \times 1$, or 1120 inch-lbs. (Art. 54, Fig. 35), or 93·3 foot-lbs.

Examples V.

1. A locomotive draws a train weighing 150 tons along a level track at 40 miles per hour, the resistances amounting to 10 lbs. per ton. What horse-power is it exerting? Find also the horse-power necessary to draw the train at the same speed (*a*) up an incline of 1 in 250, (*b*) down an incline of 1 in 250.

2. If a locomotive exerts 700 horse-power when drawing a train of 200 tons up an incline of 1 in 80 at 30 miles per hour, find the road resistances in pounds per ton.

3. A motor-car engine can exert usefully on the wheels 8 horse-power. If the car weighs 16 cwt., and the road and air resistances be taken at 20 lbs. per ton, at what speed can this car ascend a gradient of 1 in 15?

4. A winding engine draws from a coal-mine a cage which with the coal carried weighs 7 tons; the cage is drawn up 380 yards in 35 seconds. Find the average horse-power required. If the highest speed attained is 30 miles per hour, what is the horse-power exerted at that time?

5. A stream delivers 3000 cubic feet of water per minute to the highest point of a water-wheel 40 feet diameter. If 65 per cent. of the available work is usefully employed, what is the horse-power developed by the wheel?

6. A bicyclist rides up a gradient of 1 in 15 at 10 miles per hour. The

weight of rider and bicycle together is 180 lbs. If the road and other resistances are equivalent to $\frac{1}{180}$ of this weight, at what fraction of a horse-power is the cyclist working?

7. Within certain limits, the force required to stretch a spring is proportional to the amount of stretch. A spring requires a force of 800 lbs. to stretch it 5 inches: find the amount of work done in stretching it 3 inches.

8. A chain 400 feet long and weighing 10 lbs. per foot, hanging vertically, is wound up. Draw a diagram of the force required to draw it up when various amounts have been wound up from 0 to 400 feet. From this diagram calculate the work done in winding up (*a*) the first 100 feet of the chain, (*b*) the whole chain.

9. A pit cage weighing 1000 lbs. is suspended by a cable 800 feet long weighing $1\frac{3}{4}$ lbs. per foot length. How much work will be done in winding the cage up to the surface by means of the cable, which is wound on a drum?

56. It frequently happens that the different parts of a body acted upon by several forces move through different distances in the same time; an important instance is the case of the rotating parts of machines generating or transmitting power. It will be convenient to consider here the work done by forces which cause rotary motion of a body about a fixed axis.

Moment of a Force.—The moment of a force about a point is the measure of its turning effect or tendency, about that point. It is measured by the product of the force and the perpendicular distance from the point to the line of action of the force. Thus in Fig. 36, if O is a point, and AB the line of action of a force F, both in the plane of the figure, and OP is the perpendicular from O on to AB measuring r units of length, the moment of F about O is $F \times r$.

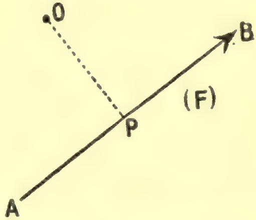


Fig. 36.

The turning tendency of F about O will be in one direction, or the opposite, according as O lies to the right or left of AB looking in the direction of the force. If O lies to the right, the moment is said to be clockwise; if to the left, contra-clockwise. In adding moments of forces about O, the clockwise and contra-clockwise moments must be taken as of opposite sign, and the

algebraic sum found. Which of the two kinds of moments is considered positive and which negative is immaterial. If O lies in the line AB, the moment of F about O is zero.[1]

The common units for the measurement of moments are pound-feet. Thus, if a force of 1 lb. has its line of action 1 foot from a fixed point, its moment about that point is one pound-foot. In Fig. 36, if the force is F *lbs.*, and OP represents r feet, the moment about O is $F \cdot r$ pound-feet.

Moment of a Force about an Axis perpendicular to its Line of Action.—If we consider a plane perpendicular to the axis and through the force, it will cut the axis in a point O; then the moment of the force about the axis is that of the force about O, the point of section of the axis by the plane. The moment of the force about the axis may therefore be defined as *the product of the force and its perpendicular distance from the axis.*

In considering the motion of a body about an axis, it is necessary to know the moments about that axis of all the forces acting on the body in planes perpendicular to the axis, whether all the forces are in the same plane or not. The total moment is called the *torque*, or twisting moment or turning moment about the axis. In finding the torque on a body about a particular axis, the moments must be added algebraically.

57. Work done by a Constant Torque or Twisting Moment.—Suppose a force F lbs. (Fig. 37) acts upon a body which turns about an axis, O, perpendicular to the line of action of F and distant r feet from it, so that the turning

[1] Note that the question whether a moment is clockwise or contra-clockwise depends upon the *aspect* of view. Fig. 36 shows a force (F) having a contra-clockwise moment about O, but this only holds for one aspect of the figure. If the force F in line AB and the point O be viewed from the other side of the plane of the figure, the moment would be called a clockwise one. This will appear clearly if the figure is held up to the light and viewed from the other side of the page. Similarly, the moment of a force about an axis will be clockwise or contra-clockwise according as the force is viewed from one end or the other of the axis. The motion of the hands of a *clock* appears *contra-clockwise* if viewed from the back through a transparent face.

moment (M) about O is F . r lb.-feet. Suppose that the force F acts successively on different parts of the body all distant r from the axis O about which it rotates, or that the force acts always on the same point C, and changes its direction as C describes its circular path about the centre O, so as to always remain tangential to this circular path; in either case the force F is always in the same direction as the displacement it is producing, and therefore the work done is equal to the product of the force and the displacement (along the circumference of the circle CDE). Let the displacement about the axis O be through an angle θ radians corresponding to an arc CD of the circle CDE, so that—

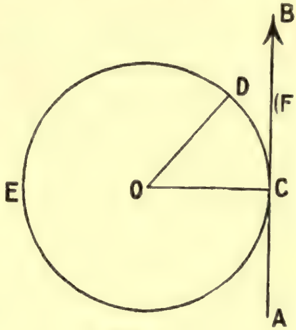


FIG. 37.

$$\frac{\text{CD}}{r} = \theta, \text{ or CD} = r \, . \, \theta$$

(The angle θ is 2π, if a displacement of one complete circuit be considered.)

The work done is F $\times$ CD = F . $r\theta$ foot-lbs.
But M = F . r lb.-feet
therefore the work done = M $\times$ θ foot-lbs.

The work done by each force is, then, the product of the turning moment and the angular displacement in radians. If the units of the turning moment are pound-feet, the work will be in foot-pounds; if the moment is in pound-inches, the work will be in inch-pounds, and so on. The same method of calculating the work done would apply to all the forces acting, and finally the total work done would be *the product of the total torque or turning moment and the angular displacement in radians.*

Again, if ω is the angular velocity in radians per second, the power or work per second is M . ω foot-lbs., and the horse-

power is $\frac{M \cdot \omega}{550}$, where M is the torque in lb.-feet; and if N is the number of rotations per minute about the axis—

$$\text{H.P.} = \frac{2\pi N \cdot M}{33{,}000}$$

This method of estimating the work done or the power, is particularly useful when the turning forces act at different distances from the axis of rotation.

We may, for purposes of calculation, look upon such a state of things as replaceable by a certain force at a certain radius, but the notion of a torque and an angular displacement seems rather less artificial, and is very useful.

The work done by a variable turning moment during a given angular displacement may be found by the method of Arts. 53 and 54. If in Figs. 33, 34, and 35 force be replaced by turning moment and space by angular displacement, the areas under the curves still represent the work done.

In twisting an elastic rod from its unstrained position the twisting moment is proportional to the angle of twist, hence the average twisting moment is half the maximum twisting moment; then, if M = maximum twisting moment, and θ = angle of twist in radians—

$$\text{the work done} = \tfrac{1}{2}M\theta$$

Example 1.—A high-speed steam-turbine shaft has exerted on it by steam jets a torque of 2100 lb.-feet. It runs at 750 rotations per minute. Find the horse-power.

The work done per minute = (torque in lb.-feet) × (angle turned through in radians)

$$= 2100 \times 750 \times 2\pi \text{ foot-lbs.}$$

$$\text{horse-power} = \frac{2100 \times 750 \times 2\pi}{33{,}000} = 300 \text{ H.P.}$$

Example 2.—An electro motor generates 5 horse-power, and runs at 750 revolutions per minute. Find the torque in pound-feet exerted on the motor spindle.

Horse-power × 33,000 = torque in lb.-feet × radians per minute

$$\text{hence torque in lb.-feet} = \frac{\text{horse-power} \times 33{,}000}{\text{radians per minute}}$$

$$= \frac{5 \times 33{,}000}{750 \times 2\pi} = 35 \text{ lb.-feet}$$

EXAMPLES VI.

1. The average turning moment on a steam-engine crankshaft is 2000 lb.-feet, and its speed is 150 revolutions per minute. Find the horse-power it transmits.

2. A shaft transmitting 50 H.P. runs at 80 revolutions per minute. Find the average twisting moment in pound-inches exerted on the shaft.

3. A steam turbine develops 250 horse-power at a speed of 200 revolutions per minute. Find the torque exerted upon the shaft by the steam.

4. How much work is required to twist a shaft through 10° if the stiffness is such that it requires a torque of 40,000 lb.-inches per radian of twist?

5. In winding up a large clock (spring) which has completely "run down," $8\frac{1}{2}$ complete turns of the key are required, and the torque applied at the finish is 200 lb.-inches. Assuming the winding effort is always proportional to the amount of winding that has taken place, how much work has to be done in winding the clock? How much is done in the last two turns?

6. A water-wheel is turned by a mean tangential force exerted by the water of half a ton at a radius of 10 feet, and makes six turns per minute. What horse-power is developed?

58. Energy.—When a body is capable of doing work, it is said to possess *energy*. It may possess energy for various reasons, such as its motion, position, temperature, chemical composition, etc.; but we shall only consider two kinds of mechanical energy.

59. A body is said to have *potential energy* when it is capable of doing work by virtue of its position. For example, when a weight is raised for a given vertical height above datum level (or zero position), it has work done upon it; this work is said to be stored as potential energy. The weight, in returning to its datum level, is capable of doing work by exerting a force (equal to its own weight) through a distance equal to the vertical height through which it was lifted, the amount of work it is capable of doing being, of course, equal to the amount of work spent in lifting it. This amount is its potential energy in its raised position, *e.g.* suppose a weight W lbs. is lifted h feet; the work is W . h foot-lbs., and the potential energy of the W lbs. is then said to be W . h foot-lbs. It is capable of doing an amount of work W . h foot-lbs. in falling.

60. Kinetic Energy is the energy which a body has in virtue of its motion.

We have seen (Art. 40) that the exertion of an unresisted force on a body gives it momentum equal to the impulse of the force. The force does work while the body is attaining the momentum, and the work so done is the measure of the kinetic energy of the body. By virtue of the momentum it possesses, the body can, in coming to rest, overcome a resisting force acting in opposition to its direction of motion, thereby doing work. The work so done is equal to the kinetic energy of the body, and therefore also to the work spent in giving the body its motion.

Suppose, as in Ex. 1, Art. 47, a body of weight W lbs. is given a velocity V feet per second by the action of a uniform force F_1 lbs. acting for t_1 seconds, and then comes to rest under a uniform resisting force F_2 lbs. in t_2 seconds. We had, in Art. 47—

$$\text{Impulse } F_1t_1 = \frac{W}{g}V = F_2t_2$$

But, the mean velocity being half the maximum under a uniform accelerating force, the distance d_1, moved in accelerating, is $\frac{1}{2}Vt_1$ feet, and that d_2, moved in coming to rest, is $\frac{1}{2}Vt_2$; hence the work done in accelerating is—

$$F_1 \times \tfrac{1}{2}Vt_1 = \frac{W}{g}V \times \tfrac{1}{2}V = \tfrac{1}{2}\frac{W}{g}V^2$$

and work done in coming to rest is—

$$F_2 \times \tfrac{1}{2}Vt_2 = \frac{W}{g}V \times \tfrac{1}{2}V = \tfrac{1}{2}\frac{W}{g}V^2$$

$$\text{hence } \tfrac{1}{2}\frac{W}{g}V^2 = F_1d_1 = F_2d_2$$

These two equalities are exactly the same as those of Ex. 1, Art. 47 $\left(\text{viz. } \frac{W}{g}V = F_1t_1 = F_2t_2\right)$, with each term multiplied by $\frac{V}{2}$, and problems which were solved from considerations of changes of momentum might often have been (alternatively) solved by considerations of change of kinetic energy.

The amount of kinetic energy possessed by a body of weight W lbs. moving at V feet per second is therefore $\frac{1}{2}\frac{W}{g}V^2$ foot-lbs.

Again, if the initial velocity had been u feet per second instead of zero, the change of momentum would have been $\frac{W}{g}(v - u)$, and we should have had—

$$F_1t_1 = \frac{W}{g}(v - u),\ v \text{ being final velocity}$$

and the work done $= F_1 \times \frac{u + v}{2} \times t_1 = \frac{W}{g}(v - u)\frac{u + v}{2}$

$$= \frac{1}{2}\frac{W}{g}(v^2 - u^2)$$

$$= \text{change of kinetic energy}$$

Similarly, in overcoming resistance at the expense of its kinetic energy, the work done by a body is equal to the change of kinetic energy whether all or only part of it is lost.

61. Principle of Work.—If a body of weight W lbs. be lifted through h feet, it has potential energy Wh foot-lbs. If it falls freely, its gain of kinetic energy at any instant is just equal to the loss of potential energy, so that the sum (potential energy) + (kinetic energy) is constant; *e.g.* suppose the weight has fallen freely x feet, its remaining potential energy is $W(h - x)$ foot-lbs. It will have acquired a velocity $\sqrt{2gx}$ feet per second (Art. 13), hence its kinetic energy $\frac{1}{2}\frac{W}{g}V^2$, will be $\frac{1}{2}\frac{W}{g} \times 2gx = Wx$ foot-lbs., hence $W(h-x)+\frac{1}{2}\frac{W}{g}V^2 = Wh$, which is independent of the value of x, and no energy has been lost.

Note that for a particular system of bodies the sum of potential and kinetic energies is generally *not* constant. Thus, *although momentum is conservative, mechanical energy is not.* For example, when a body in motion is brought to rest by a resisting force of a frictional kind, *mechanical energy* is lost. The energy appears in other forms, chiefly that of heat.

Principle of Work.—Further, if certain forces act upon

a body, doing work, and other forces, such as frictional ones, simultaneously resist the motion of the body, the excess of the work done by the urging forces over that done against the resistances gives the kinetic energy stored in the body. Or we may deduct the resisting forces from the urging forces at every instant, and say that the work done by the effective or net accelerating forces is equal to the kinetic energy stored. Thus in Fig. 38, representing the forces and work done graphically as in Art. 54, if the ordinates of the curve MP represent the forces urging the body forward, and the ordinates of M′P′ represent the resistances to the same scale, the area MPNO represents the work done; the work lost against resistances is represented by the area M′P′NO, and the difference between these two areas, viz. the area MPP′M′, represents the kinetic energy stored during the time that the distance ON has been traversed. If the body was at rest at position O, MPP′M′ represents the total kinetic energy, and if not, its previous kinetic energy must be added to obtain the total stored at the position ON. From a diagram, such as Fig. 38, the velocity

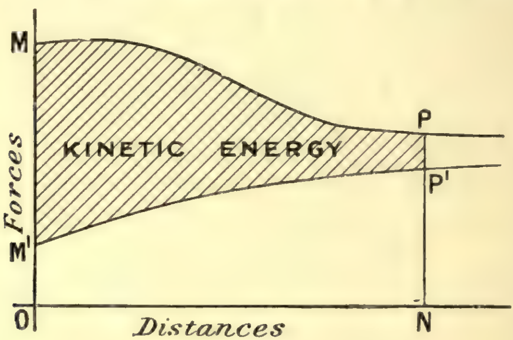


Fig. 38.

can be obtained, if the mass of the moving body is known, by the relation, kinetic energy $= \frac{1}{2}(\text{mass}) \times (\text{velocity})^2$.

Fig. 39 illustrates the case of a body starting from rest and coming to rest again after a distance O*g*, such, for example, as an electric car between two stopping-places. The driving forces proportional to the ordinates of the curve *abec* cease

after a distance *oc* has been traversed, and (by brakes) the resisting forces proportional to the ordinates of the curve *def* increase. The area *abed* represents the kinetic energy of the car after a distance *oc*, and the area *efgc* represents the work

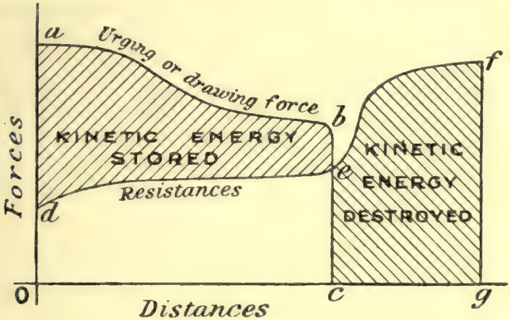


FIG. 39.

done by the excess of resisting force over driving force. When the latter area is equal to the former, the car will have come to rest.

The kinetic energy which a body possesses in virtue of its rotation about an axis will be considered in a subsequent chapter.

Example 1.—Find the work done by the charge on a projectile weighing 800 lbs., which leaves the mouth of a cannon at a velocity of 1800 feet per second. What is the kinetic energy of the gun at the instant it begins to recoil if its weight is 25 tons?

The work done is equal to the kinetic energy of the projectile—

$$\text{K.E.} = \frac{1}{2} \times \frac{W}{g} \times V^2 = \frac{1}{2} \times \frac{800}{32 \cdot 2} \times (1800)^2 = 40{,}200{,}000 \text{ foot-lbs.}$$

The momentum of the gun being equal to that of the projectile, the velocity of the gun is—

$$1800 \times \frac{800}{25 \times 2240} = 25 \cdot 71 \text{ feet per second}$$

$$\text{and the K.E.} = \frac{1}{2} \times \frac{25 \times 2240}{32 \cdot 2} \times (25 \cdot 71)^2 = 577{,}000 \text{ foot-lbs.}$$

It may be noticed that the kinetic energies of the projectile

and cannon are inversely proportional to their weights. The K.E. is $\frac{1}{2} \times \frac{W}{g} \times V^2$, or $\frac{1}{2} \times \frac{W}{g} \times V \times V$, which is $\frac{1}{2} \times$ momentum $\times$ velocity. The momentum of the gun and that of the projectile are the same (Art. 52), and therefore their velocities are inversely proportional to their weights; and therefore the products of velocities and half this momentum are inversely proportional to their respective weights.

Example 2.—A bullet weighing 1 oz., and moving at a velocity of 1500 feet per second, overtakes a block of wood moving at 40 feet per second and weighing 5 lbs. The bullet becomes embedded in the wood without causing any rotation. Find the velocity of the wood after the impact, and how much kinetic energy has been lost.

Let V = velocity of bullet and block after impact.

$$\text{Momentum of bullet} = \frac{1}{16} \times \frac{1500}{g} = \frac{93.75}{g}$$

$$\text{momentum of block} = \frac{5}{g} \times 40 = \frac{200}{g}$$

$$\left.\begin{array}{r}\text{hence total momentum before} \\ \text{and after impact}\end{array}\right\} = \frac{293.75}{g}$$

$$\text{Total momentum after impact} = \frac{5\frac{1}{16}}{g} \times V = \frac{293.75}{g}$$

$$\text{and therefore } V = \frac{293.75}{5.0625} = 58.1 \text{ feet per second}$$

$$\text{Kinetic energy of bullet} = \frac{1}{2} \times \frac{1}{16} \times \frac{1}{32.2} \times 1500 \times 1500 = 2183 \text{ foot-lbs.}$$

$$\text{Kinetic energy of block} = \frac{1}{2} \times \frac{5}{32.2} \times 40 \times 40 = 124 \quad ,,$$

$$\text{Total K.E. before impact} = 2307 \quad ,,$$

$$\text{Total K.E. after impact} = \frac{1}{2} \times \frac{5.0625}{32.2} \times 58.1 \times 58.1 = 265 \text{ foot-lbs.}$$

$$\text{Loss of K.E. at impact} = 2307 - 265 = 2042 \quad ,,$$

Example 3.—A car weighs 12·88 tons, and starts from rest; the resistance of the rails may be taken as constant and equal to 500 lbs. After it has moved S feet from rest, the tractive force, F lbs., exerted by the motors is as follows:—

S ...	0	20	50	80	110	130	160	190	200
F ...	1280	1270	1220	1110	905	800	720	670	660

Find the velocity of the car after it has gone 200 feet from rest; also find the velocity at various intermediate points, and plot a curve of velocity on a base of space described.

Plot the curve of F and S as in Fig. 40, and read off the force every 20 feet, say, starting from S = 10, and subtract 500 lbs. resistance from each, as follows:—

S ...	10	30	50	70	90	110	130	150	170	190
F ...	1275	1260	1220	1150	1050	905	800	740	695	670
F−500	775	760	720	650	550	405	300	240	195	170

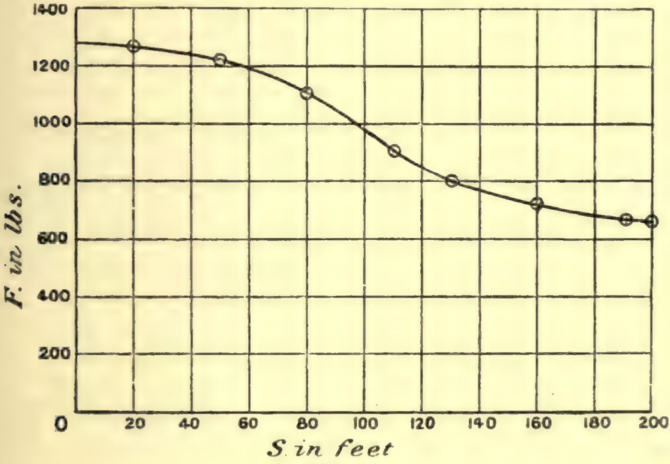


FIG. 40.

The mean accelerating force during the first 20 feet of motion is approximately equal to that at S = 10, viz. 775 lbs.; hence the work stored as kinetic energy (K.E.), *i.e.* the gross work done less that spent against resistance, is—

(1275 × 20) − (500 × 20), or 775 × 20 foot-lbs. = 15,500 foot-lbs.

Then, if V is the velocity after covering S feet, for S = 20—

$$\text{K.E.} = \frac{1}{2} \times \frac{W}{g}V^2 = 15{,}500$$

$$\text{and } W = 12{\cdot}88 \times 2240 \text{ lbs.}$$

therefore $\frac{W}{g}$, the mass of the car is $\frac{12{\cdot}88 \times 2240}{32{\cdot}2}$ or 896 units, and—

$$\frac{1}{2} \times \frac{W}{g} V^2 = \frac{1}{2} \times 896 \times V^2 = 15{,}500$$

$$V^2 = \frac{15{,}500}{448} = 34{\cdot}8$$

$$V = \sqrt{34{\cdot}8} = 5{\cdot}90 \text{ feet per second}$$

Similarly, finding the gain of kinetic energy in each 20 feet, the square of velocity (V^2), and the velocity V, we have from S = 20 to S = 40—

gain of K.E. = 760 × 20 = 15,200 foot-lbs.

∴ total K.E. at S = 40 is

15,500 + 15,200 = 30,700 foot-lbs.

and so on, thus—

S	0	20	40	60	80	100	120	140	160	180	200
Gain of K.E. in 20 feet, foot-lbs.	0	15500	15200	14400	13000	11000	8100	6000	4800	3900	3400
Total K.E., foot-lbs.	0	15500	30700	45100	58100	69100	77200	83200	88000	91900	95300
V^2 or $\frac{\text{K.E.}}{448}$	0	34·8	68·5	100·6	129·4	154·0	172·1	185·5	196·2	204·8	212·5
V ft. per sec.	0	5·90	8·28	10·03	11·34	12·40	13·12	13·62	14·01	14·30	14·58

These velocities have been plotted on a base of spaces in Fig. 41.

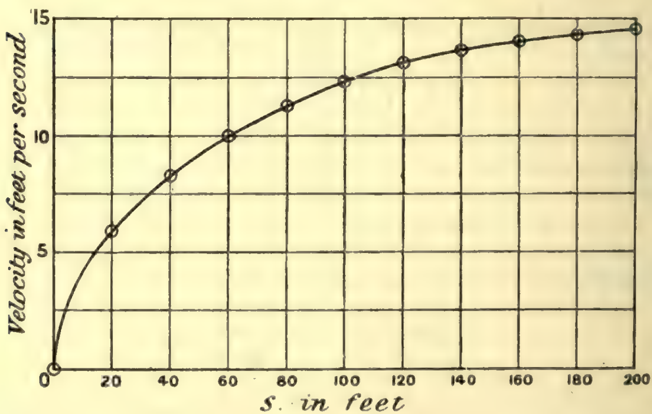


FIG. 41.

Example 4.—From the results of Example 3, find in what time the car travels the distance of 20 feet from S = 80 to S = 100, and draw a curve showing the space described up to any instant during the time in which it travels the first 200 feet.

$$\text{At } S = 80,\ V = 11{\cdot}34 \text{ feet per second}$$
$$\text{at } S = 100,\ V = 12{\cdot}40 \text{ feet per second}$$

hence the mean velocity for such a short interval may be taken as approximately—

$$\frac{11{\cdot}34 + 12{\cdot}40}{2}, \text{ or } 11{\cdot}87 \text{ feet per second}$$

Hence the time taken from S = 80 to S = 100 is approximately—

$$\frac{20}{11{\cdot}87} = 1{\cdot}685 \text{ seconds}$$

Similarly, we may find the time taken to cover each 20 feet, and so find the total time occupied, by using the results of Ex. 3, as follows. The curve in Fig. 42 has been plotted from these numbers.

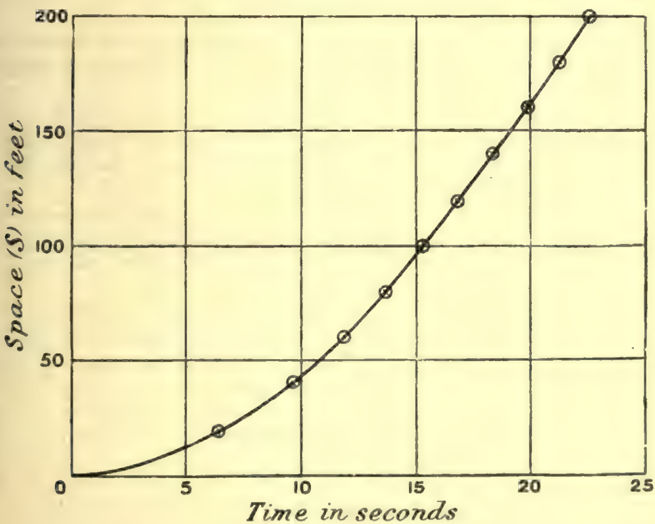


FIG. 42.

S	0	20	40	60	80	100	120	140	160	180	2
Mean velocity for last 20ft., feet per sec.	0	2·95	7·09	9·15	10·68	11·87	12·76	13·37	13·81	14·15	14·
Time for last 20 feet, seconds	0	6·780	2·824	2·188	1·872	1·685	1·568	1·496	1·447	1·413	1·3
Total time, t seconds	0	6·780	9·604	11·792	13·664	15·349	16·917	18·413	19·850	21·263	22·6

EXAMPLES VII.

1. Find in foot-pounds the kinetic energy of a projectile weighing 800 lbs. moving at 1000 feet per second. If it is brought to rest in 3 feet, find the space average of the resisting force.

2. At what velocity must a body weighing 5 lbs. be moving in order to have stored in it 60 foot-lbs. of energy?

3. What is the kinetic energy in inch-pounds of a bullet weighing 1 oz. travelling at 1800 feet per second? If it is fired directly into a suspended block of wood weighing 1·25 lb., how much kinetic energy is lost in the impact?

4. A machine-gun fires 300 bullets per minute, each bullet weighing 1 oz. and having a muzzle velocity of 1700 feet per second. At what average horse-power is the gun working?

5. A jet of water issues in a parallel stream at 90 feet per second from a round nozzle 1 inch in diameter. What is the horse-power of the jet? One cubic foot of water weighs 62·5 lbs.

6. Steam to drive a steam impact turbine issues in a parallel stream from a jet $\frac{1}{4}$ inch diameter at a velocity of 2717 feet per second, and the density of the steam is such that it occupies 26·5 cubic feet per pound. Find the horse-power of the jet.

7. A car weighing 10 tons attains a speed of 15 miles per hour from rest in 24 seconds, during which it covers 100 yards. If the space-average of the resistances is 30 lbs. per ton, find the average horse-power used to drive the car.

8. How long will it take a car weighing 11 tons to accelerate from 10 miles per hour to 15 miles per hour against a resistance of 25 lbs. per ton, if the motors exert a uniform tractive force on the wheels and the horse-power is 25 at the beginning of this period?

9. A car weighing 12 tons is observed to have the following tractive forces F lbs. exerted upon it after it has travelled S feet from rest :—

S ...	0	10	30	50	65	80	94	100
F ...	1440	1390	1250	1060	910	805	760	740

The constant resistance of the road is equivalent to 600 lbs. Find the velocity of the car after it has covered 100 feet. Plot a curve showing the velocity at all distances for 100 feet from the starting-point. What is the space-average of the effective or accelerating force on the car?

10. From the results of the last question plot a curve showing the space described at any instant during the time taken to cover the first 100 feet. How long does the car take to cover 100 feet?

11. A machine having all its parts in rigid connection has 70,000 foot-pounds of kinetic energy when its main spindle is making 49 rotations per minute. How much extra energy will it store in increasing its speed to 50 rotations per minute?

12. A machine stores 10,050 foot-lbs. of kinetic energy when the speed of its driving-pulley rises from 100 to 101 revolutions per minute. How much kinetic energy would it have stored in it when its driving-pulley is making 100 revolutions per minute?

CHAPTER IV

MOTION IN A CIRCLE: SIMPLE HARMONIC MOTION

62. Uniform Circular Motion.—Suppose a particle describes about a centre O (Fig. 43), a circle of radius r feet with uniform angular velocity ω radians per second. Then its velocity, v, at any instant is of magnitude ωr (Art. 33), and its direction is along the tangent to the circle from the point in the circumference which it occupies at that instant. Although its velocity is always of magnitude ωr, its direction changes. Consider the change in velocity between two points, P and Q, on its path at an angular distance θ apart (Fig. 43). Let the vector cb parallel to the tangent PT represent the linear velocity v at P, and let the vector ab, of equal length to cb and parallel to QT′, the tangent at Q, represent the linear velocity v at Q. Then, to find the change of velocity between P and Q, we must subtract the velocity at P from that at Q; in vectors—

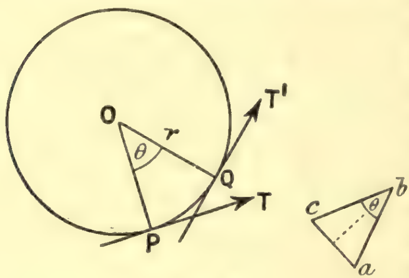


FIG. 43.

$$ab - cb = ab + bc = ac \text{ (Art. 27)}$$

Then the vector ac represents the change of velocity between the positions P and Q. Now, since $\hat{abc} = \hat{POQ} = \theta$, length

$ac = 2ab \cdot \sin\frac{\theta}{2}$, which represents $2v\sin\cdot\frac{\theta}{2}$, and the time taken between the positions P and Q is $\frac{\theta}{\omega}$ seconds (Art. 33).

Therefore the average change of velocity per second is—

$$2\,v\sin\frac{\theta}{2} \div \frac{\theta}{\omega} \text{ or } \omega v \cdot \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}$$

which is the average acceleration. Now, suppose that Q is taken indefinitely close to P—that is, that the angle θ is indefinitely reduced; then the ratio $\dfrac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}$ has a limiting value unity, and the average change of velocity per second, or average acceleration during an indefinitely short interval is ωv, or $\omega^2 r$ or $\frac{v^2}{r}$, since $v = \omega r$. This average acceleration during an indefinitely reduced interval is what we have defined (Art. 9) as actual acceleration, so that the acceleration at P is $\omega^2 r$ or $\frac{v^2}{r}$ feet per second per second. And as the angle θ is diminished indefinitely and Q thereby approaches P, the vector ab, remaining of the same length, approaches cb (a and c being always equidistant from b), and the angle $b\hat{c}a$ increases and approaches a right angle as θ approaches zero. Ultimately the acceleration ($\omega^2 r$) is perpendicular to PT, the tangent at P, *i.e.* it is towards O.

63. Centripetal and Centrifugal Force. — In the previous article we have seen that if a small body is describing a circle of radius r feet about a centre O with angular velocity ω radians per second, it must have an acceleration $\omega^2 r$ towards O; hence the force acting upon it must be directed *towards* the centre O and of magnitude equal to its (mass) $\times\ \omega^2 r$ or $\frac{W}{g}\omega^2 r$ lbs., where W is its weight in pounds This force causing

the circular motion of the body is sometimes called the centripetal force. There is (Art. 51), by the third law of motion, a reaction of equal magnitude upon the medium which exerts this centripetal force, and this reaction is called the *centrifugal force*. It is directed away from the centre O, and is exerted upon the matter which impresses the equal force $\frac{W}{g}\omega^2 r$ upon the revolving body; it is not to be reckoned as a force acting upon the body describing a circular path.

A concrete example will make this clear. If a stone of weight W lbs. attached to one end of a string r feet long describes a horizontal circle with constant angular velocity ω radians per second, and is supported in a vertical direction by a smooth table, so that the string remains horizontal, the force which the string exerts upon the stone is $\frac{W}{g}\omega^2 r$ *towards* the centre of the circle. The stone, on the other hand, exerts on the string an *outward* pull $\frac{W}{g}\omega^2 r$ away from the centre. In other cases of circular motion the inward centripetal force may be supplied by a thrust instead of a tension; *e.g.* in the case of a railway carriage going round a curved line, the centripetal thrust is supplied by the rail, and the centrifugal force is exerted outward on the rail by the train.

64. Motion on a Curved "Banked" Track.—Suppose a body, P (Fig. 44), is moving with uniform velocity, v, round a

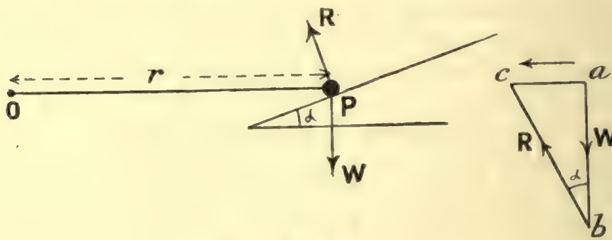


FIG. 44.

smooth circular track of radius OP equal to r feet. At what angle to the horizontal plane shall the track be inclined or

"banked" in order that the body shall keep in its circular path?

There are two forces acting on the body—(1) its own weight, W; (2) the reaction R of the track which is perpendicular to the smooth track. These two have a horizontal resultant $\frac{W}{g} \cdot \frac{v^2}{r}$ towards the centre O of the horizontal circle in which the body moves. If we draw a vector, *ab* (Fig. 44), vertically, to represent W, then R is inclined at an angle α to it, where α is the angle of banking of the track. If a vector, *bc*, be drawn from *b* inclined at an angle α to *ab*, to meet *ac*, the perpendicular to *ab* from *a*, then *bc* represents R, and *ac* or $(ab + bc)$ represents the resultant of W and R, viz. $\frac{W}{g} \cdot \frac{v^2}{r}$, and—

$$\tan \alpha = \frac{ac}{ab} = \frac{W}{g} \cdot \frac{v^2}{r} \div W = \frac{v^2}{gr}$$

which gives the angle α required.

65. Railway Curves.—If the lines of a railway curve be laid at the same level, the centripetal thrust of the rails on the wheels of trains would act on the flanges of the wheels, and the centrifugal thrust of the wheel on the track would tend to push it sideways out of its place. In order to have the action and reaction normal to the track the outer rail is raised, and the track thereby inclined to the horizontal. The amount of this "superelevation" suitable to a given speed is easily calculated.

Let G be the gauge in inches, say, v the velocity in feet per second, and r the radius of the curve in feet. Let AB (Fig. 45) represent G; then AC represents the height in inches (exaggerated) which B stands above A, and $\hat{ABC}$ is the angle of banking, as in Art. 64. Then AC = AB $\sin \alpha$ = AB $\tan \alpha$ nearly, since α is always very small; hence, by Art. 64, AC represents G $\tan \alpha$, or $G\frac{v^2}{gr}$ inches.

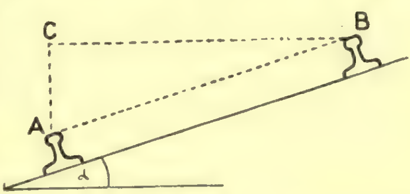


FIG. 45.

66. Conical Pendulum.—This name is applied to a combination consisting of a small weight fastened to one end of a string, the other end of which is attached to a fixed point, when the weight keeping the string taut, describes a horizontal circle about a centre vertically under the fixed point.

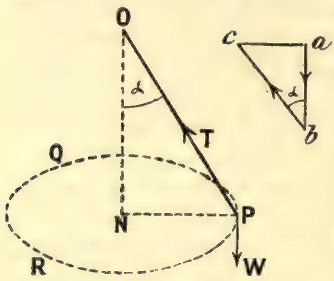


Fig. 46.

Fig. 46 represents a conical pendulum, where a particle, P, attached by a thread to a fixed point, O, describes the horizontal circle PQR with constant angular velocity about the centre N vertically under O.

Let T = tension of the string OP in lbs.;
ω = angular velocity of P about N in radians per second;
W = weight of particle P in lbs.;
r = radius NP of circle PQR in feet;
l = length of string OP in feet;
α = angle which OP makes with ON, viz. $\mathrm{P\hat{O}N}$;
h = height ON in feet;
g = acceleration of gravity in feet per second per second.

At the position shown in Fig. 46 P is acted upon by *two* forces—(1) its own weight, W; (2) the tension T of the string OP. These have a resultant in the line PN (towards N), the vector diagram being set off as in Art. 64, *ab* vertical, representing the weight W, of P, and *bc* the tension T. Then the vector $ac = ab + bc$, and represents the resultant force $\frac{W}{g} \times \omega^2 r$ along PN; hence—

$$\tan \alpha = \frac{ac}{ab} = \frac{W}{g}\omega^2 r \div W = \frac{\omega^2 r}{g}$$

$$\text{Also ON or } h = \text{NP} \div \tan \alpha = r \div \frac{\omega^2 r}{g} = \frac{g}{\omega^2} \text{ feet}$$

hence the height h of the conical pendulum is dependent only on the angular velocity about N, being inversely proportional to the square of that quantity.

Since h or $l \cos \alpha = \frac{g}{\omega^2}$, $\omega^2 = \frac{g}{h}$ and $\omega = \sqrt{\frac{g}{h}}$

Also the time of one complete revolution of the pendulum is—

$$\frac{\text{angle in a circle}}{\text{angular velocity}} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$$

the period of revolution being proportional to the square root of the height of the pendulum, and the number of revolutions per minute being therefore inversely proportional to the square root of the height. This principle is made use of in steam-engine governors, where a change in speed, altering the height of a modified conical pendulum, is made to regulate the steam supply.

67. Motion in a Vertical Circle.—Suppose a particle or small body to move, say, contra-clockwise in a vertical circle with centre O (Fig. 47). It may be kept in the circular path by a string attached to O, or by an inward pressure of a circular track. Taking the latter instance—

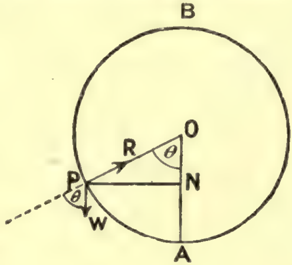


FIG. 47.

Let R = the normal inward pressure of the track;
W = the weight of the rotating body in pounds;
v = its velocity in feet per second in any position P such that OP makes an angle θ to the vertical OA, A being the lowest point on the circumference;
v_A = the velocity at A;
r = the radius of the circle in feet.

Then the kinetic energy at A is $\frac{1}{2}\frac{W}{g}v_A^2$

At P the potential energy is W × AN, and the kinetic energy is $\frac{1}{2}\frac{W}{g}v^2$, and since there is no work done or lost between A and P, the total mechanical energy at P is equal to that at A (Art. 61). Therefore—

$$\frac{1}{2}\frac{W}{g}\cdot v^2 + W\,.\,AN = \frac{1}{2}\frac{W}{g}v_A{}^2$$

$$\text{hence } v^2 + 2g\,.\,AN = v_A{}^2 \quad . \quad . \quad . \quad . \quad (1)$$

Neglecting gravity, the motion in a circle would be uniform, and would cause a reaction $\frac{W}{g}\cdot\frac{v^2}{r}$ from the track (Art. 63). And in addition the weight has a component W cos θ in the direction OP, which increases the inward reaction of the track by that amount; hence the total normal pressure—

$$R = \frac{W}{g}\cdot\frac{v^2}{r} + W\cos\theta \quad . \quad . \quad . \quad . \quad (2)$$

The value of R at any given point can be found by substituting for v from equation (1) provided v_A is known. The least value of R will be at B, the highest point of the circle, where gravity diminishes it most. If v_A is not sufficient to make R greater than zero for position B, the particle will not describe a complete circle. Examining such a case, the condition, in order that a complete revolution may be made without change in the sign of R, is—

$$R_B > 0$$

$$\textit{i.e.}\ \frac{W}{g}\cdot\frac{v_B{}^2}{r} + W\cos 180^\circ > 0$$

or, since cos $180^\circ = -1$—

$$\frac{W}{g}\cdot\frac{v_B{}^2}{r} > W$$

$$\text{or } v_B{}^2 > gr$$

and since $v_B{}^2 = v_A{}^2 - 2g\,.\,AB = v_A{}^2 - 4gr$, substituting for $v_B{}^2$, the condition is—

$$v_A{}^2 - 4gr > gr$$

$$v_A{}^2 > 5gr$$

$$v_A > \sqrt{5gr}$$

i.e. the velocity at A must be greater than that due to falling through a height $\frac{5}{2}r$, for which the velocity would be $\sqrt{5gr}$ (Art. 28). For example, in a centrifugal railway ("looping the loop") the necessary velocity on entering the track at the

lowest point, making no allowance for frictional resistances, may be obtained by running down an incline of height greater than two and a half times the radius of the circular track.

If the centripetal force is capable of changing sign, as in the case of the pressure of a tubular track, or the force in a light stiff radius rod supporting the revolving weight, the condition that the body shall make complete revolutions is that v_B shall be greater than zero, and since $v_B^2 = v_A^2 - 4gr$, the condition is—

$$v_A^2 > 4gr$$
$$v_A > \sqrt{4gr}$$

i.e. the velocity at A shall be greater than that due to falling through a height equal to the diameter of the circle. Similarly, the position at which the body will cease to describe a circular track (in a forward direction) if v_A is too small for a complete circuit, when the force can change sign and when it can not, may be investigated by applying equations (1) and (2), which will also give the value of R for any position of the body.

The pendulum bob, suspended by a thread, is of course limited to oscillation of less than a semicircle or to complete circles.

Example 1.—At what speed will a locomotive, going round a curve of 1000-feet radius, exert a horizontal thrust on the outside rail equal to $\frac{1}{100}$ of its own weight?

Let W = the weight of loco,

v = its velocity in feet per second.

$$\text{Centrifugal thrust} = \frac{W}{g} \cdot \frac{v^2}{1000} = \frac{1}{100}W$$

$$\therefore v^2 = \frac{1000 \times g}{100} = 322$$

$v = 17{\cdot}95$ feet per second, equivalent to $12{\cdot}22$ miles per hour

Example 2.—A uniform disc rotates 250 times per minute about an axis through its centre and perpendicular to its plane. It has attached to it two weights, one of 5 lbs. and the other of 7 lbs., at an angular distance of 90° apart, the first being 1 foot and the second 2 feet from the axis. Find the magnitude and direction of the resultant centrifugal force on the axis. Find, also,

where a weight of 12 lbs. must be placed on the disc to make the resultant centrifugal force zero.

The angular velocity is $\frac{250 \times 2\pi}{60} = \frac{25\pi}{3}$ radians per second

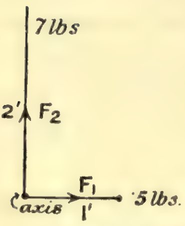


Fig. 48.

$$\left.\begin{array}{l}\text{The centrifugal pull } F_1 \text{ (Fig. 48) is} \\ \text{then } \frac{5}{32{\cdot}2} \times \left(\frac{25\pi}{3}\right)^2 \times 1\end{array}\right\} = 106 \text{ lbs.}$$

$$\left.\begin{array}{l}\text{and the centrifugal pull } F_2 \text{ is} \\ \frac{7}{32{\cdot}2} \times \left(\frac{25\pi}{3}\right)^2 \times 2\end{array}\right\} = 297 \text{ lbs.}$$

hence the resultant R of F_1 and F_2 at right angles is—

$$R = \sqrt{106^2 + 297^2} = 315 \text{ lbs.}$$

at an angle $\tan^{-1}\frac{106}{297} = \tan^{-1} 0{\cdot}357 = 19{\cdot}6°$ to the direction of F_2 (Arts. 24 and 44)

To neutralize this, a force of 315 lbs. will be required in the opposite direction.

Let x = radius in feet of the 12-lbs. weight placed at 180 − 19·6 or 160·4° contra-clockwise from F_2.

$$\text{Then } \frac{12}{32{\cdot}2} \times \left(\frac{25\pi}{3}\right)^2 \times x = 315$$

$$\text{hence } x = 1{\cdot}23 \text{ feet}$$

Example 3.—Find in inches the change in height of a conical pendulum making 80 revolutions per minute when the speed *increases two per cent.*

The increase in speed is $\frac{2}{100} \times 80 = 1{\cdot}6$ revolutions per minute to 81·6 revolutions per minute.

The height is $\frac{g}{\omega^2}$ (Art. 66), where ω is the angular velocity in radians per second.

At 80 revolutions per minute the angular velocity is—

$$\frac{2\pi \times 80}{60} = \frac{8\pi}{3} \text{ radians per second}$$

$$\text{hence the height } h_{80} = \frac{g}{\omega^2} = \frac{32{\cdot}2 \times 9}{64\pi^2}$$

$$= 0{\cdot}4585 \text{ foot}$$

At 81·6 revolutions per minute the angular velocity is—

$$\frac{2\pi \times 81\cdot6}{60} = \frac{8\cdot16\pi}{3} \text{ radians per second}$$

$$\text{and the height is } h_{81\cdot6} = \frac{32\cdot2 \times 9}{66\cdot6\pi^2} = 0\cdot4411 \text{ foot}$$

$$\left.\begin{array}{l}\text{hence the } \textit{decrease} \text{ in height is} \\ 0\cdot4585 - 0\cdot4411\end{array}\right\} = 0\cdot0174 \text{ foot or } 0\cdot207 \text{ inch}$$

Example 4.—A piece of lead is fastened to the end of a string 2 feet long, the other end of which is attached to a fixed point. With what velocity must the lead be projected in order to describe a horizontal circle of 2 feet diameter?

Let OP, Fig. 49, represent the string; then the horizontal line PN is to be 1 foot radius.

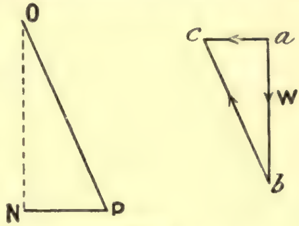


FIG. 49.

In the vector triangle *abc*, *ab* represents W, the weight of lead, *bc* the tension T of the string OP, and *ac* their resultant; then—

$$\frac{NP}{ON} = \frac{ac}{ab} = \frac{W}{g} \cdot \frac{v^2}{r} \div W = \frac{v^2}{g \times 1}$$

where v = velocity in feet per second;

$$\text{hence } v^2 = g \times \frac{NP}{ON} = g \times \frac{1}{\sqrt{3}} = \frac{32\cdot2}{\sqrt{3}} = 18\cdot57$$

$$\text{and } v = 4\cdot309 \text{ feet per second}$$

Exercise 5.—A stone weighing $\frac{1}{4}$ lb. is whirling in a vertical circle at the extremity of a string 3 feet long. Find the velocity of the stone and tension of the string—(1) at the highest position, (2) at lowest, (3) midway between, if the velocity is the least possible for a complete circle to be described.

If the velocity is the least possible, the string will just be slack when the stone is at the highest point of the circle.

Let v_0 be the velocity at the highest point, where the weight just supplies the centripetal force;

$$\text{(1) Then } \frac{1}{4} \times \frac{1}{32\cdot2} \times \frac{v_0^2}{3} = \frac{1}{4}$$

$$v_0^2 = 3 \times 32\cdot2 = 96\cdot6$$

$$\text{and } v_0 = 9\cdot83 \text{ feet per second.}$$

(2) At the lowest point let the velocity be v_1 feet per second.

Since there is no loss of mechanical energy, the gain of kinetic energy is $\frac{1}{4} \times 6$ foot-lbs., hence—

$$\frac{1}{2} \cdot \frac{1}{4} \times \frac{1}{g} \times v_1^2 = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{g} v_0^2 + \frac{1}{4} \cdot 6$$

$$\text{and } v_1^2 = v_0^2 + 2 \cdot g \cdot 6$$

$$= 96{\cdot}6 + 386{\cdot}4 = 483 \text{ (or } 5 \times g \times 3)$$

$$v_1 = \sqrt{483} = 22 \text{ feet per second (nearly)}$$

and the tension is $\left.\frac{1}{4} + \frac{1}{4} \cdot \frac{1}{32{\cdot}2} \cdot \frac{483}{3}\right\} = \frac{1}{4} + \frac{161}{128{\cdot}8} = 1{\cdot}5$ lbs., or six times the weight of the stone

(3) When the string is horizontal, if v' = velocity in feet per second—

$$\text{similarly, } \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{g} v'^2 = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{g} v_0^2 + \frac{1}{4} \cdot 3$$

$$v'^2 = v_0^2 + 2g \times 3$$

$$= 96{\cdot}6 + 193{\cdot}2$$

$$v' = \sqrt{289{\cdot}8} = 17 \text{ feet per second}$$

and the tension is $\left.\frac{1}{4} \cdot \frac{1}{32{\cdot}2} \times \frac{289{\cdot}8}{3}\right\} = 0{\cdot}75$ lb., or three times the weight of the stone

Examples VIII.

1. How many circuits per minute must a stone weighing 4 ozs. make when whirled about in a horizontal circle at the extremity of a string 5 feet long, in order to cause a tension of 2 lbs. in the string?

2. At what speed will a locomotive produce a side thrust equal to $\frac{1}{80}$ of its own weight on the outer rail of a level curved railway line, the radius of the curve being 750 feet?

3. What is the least radius of curve round which a truck may run on level lines at 20 miles per hour without producing a side thrust of more than $\frac{1}{100}$ of its own weight?

4. How much must the outer rail of a line of 4 feet $8\frac{1}{2}$ inches gauge be elevated on a curve of 800 feet radius in order that a train may exert a thrust normal to the track when travelling at 30 miles per hour?

5. The outer rail of a pair, of 4 feet $8\frac{1}{2}$ inches gauge, is elevated $2\frac{1}{2}$ inches, and a train running at 45 miles per hour has no thrust on the flanges of either set of wheels. What is the radius of the curve?

6. At what speed can a train run round a curve of 1000 feet radius without having any thrust on the wheel flanges when the outer rail is laid 1·5 inches above the inner one, and the gauge is 4 feet $8\frac{1}{2}$ inches?

7. To what angle should a circular cycle-track of 15 laps to the mile be

banked for riding upon at a speed of 30 miles per hour, making no allowance for support from friction?

8. A string 3 feet long, fixed at one end, has attached to its other end a stone which describes a horizontal circle, making 40 circuits per minute. What is the inclination of the string to the vertical? What is its tension?

9. What percentage change of angular speed in a conical pendulum will correspond to the decrease in height of 3 per cent.?

10. The revolving ball of a conical pendulum weighs 5 lbs., and the height of the pendulum is 8 inches. What is its speed? If the ball is acted upon by a vertical downward force of 1 lb., what is then its speed when its height is 8 inches? Also what would be its speed in the case of a vertical upward force of 1 lb. acting on the ball?

11. What will be the inclination to the vertical of a string carrying a weight suspended from the roof of a railway carriage of a train going round a curve of 1000 feet radius at 40 miles per hour?

12. A body weighing $\frac{1}{2}$ lb., attached to a string, is moving in a vertical circle of 6 feet diameter. If its velocity, when passing through the lowest point, is 40 feet per second, find its velocity and the tension of the string when it is 2 feet and when it is 5 feet above the lowest point.

68. Simple Harmonic Motion.—This is the simplest type of reciprocating motion. If a point Q (Fig. 50) describes a circle AQB with constant angular velocity, and P be the rectangular projection of Q on a fixed diameter AB of the circle, then the oscillation to and fro of P along AB is defined as Simple Harmonic Motion.

Let the length OA of the radius be a feet, called the amplitude of oscillation.

Let ω be the angular velocity of Q in radians per second.

Let θ be the angle $A\hat{O}Q$ in radians, denoting any position of Q.

Suppose the motion of Q to be, say, contra-clockwise.

A complete vibration or oscillation of P is reckoned in this country as the path described by P whilst Q describes a complete circle.

Let T = the period in seconds of one complete vibration; then, since this is the same as that for one complete circuit made by Q—

$$T = \frac{\text{radians in one circle}}{\text{radians described per second}} = \frac{2\pi}{\omega} \quad . \quad (1)$$

Let x = distance OP of P from O in feet, reckoned positive towards A, then $x = a \cos \theta$;

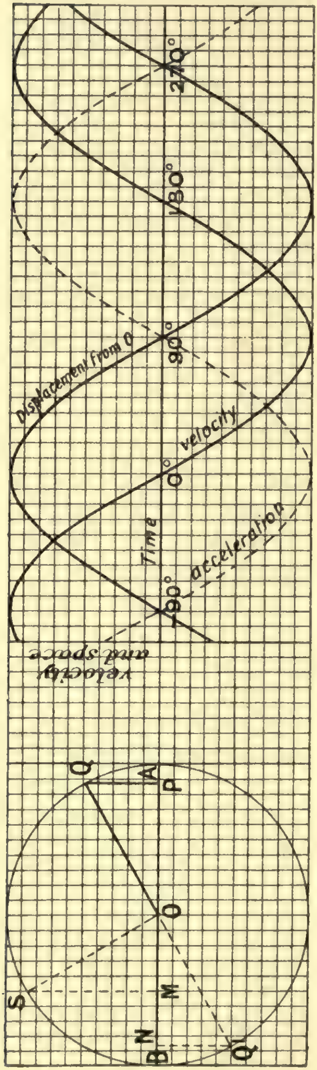


FIG. 50.

and let v = velocity of P in feet per second in position θ.

Draw OS perpendicular to OQ to meet the circumference of the circle AQSB in S, and draw SM perpendicular to AB to meet it in M.

Then for the position or phase shown in the figure, the velocity of Q is ωa (Art. 33) in the direction perpendicular to OQ, *i.e.* parallel to OS. Resolving this velocity along the diameter AB, OSM being a vector triangle, the component velocity of Q parallel to AB is $\frac{\text{OM}}{\text{OS}} \times \omega a$, or $\omega a \sin \theta$, or $\omega \,.\, \text{OM}$. This is then the velocity of P towards O, the mid-path.

$$\text{Since } \sin \theta = \frac{\text{OM}}{\text{OS}}$$

$$= \frac{\sqrt{a^2 - x^2}}{a}$$

$$v = \omega a \sin \theta$$

$$= \omega \sqrt{a^2 - x^2}$$

which gives the velocity of P in terms of the amplitude and position.

Or, if OS represents geometrically the velocity of Q, then OM represents that of P to the same scale.

Acceleration of P.—The acceleration of Q is $\omega^2 a$ along QO towards O (Art. 62). Resolving this acceleration, the component in direction AB is $\omega^2 a \times \frac{PO}{QO}$, or $\omega^2 a . \cos \theta$, or $\omega^2 . x$, towards O; and it should be noted that at unit distance from O, when $x = 1$ foot, the acceleration of P is ω^2 feet per second per second.

The law of acceleration of a body having simple harmonic motion, then, is, that the acceleration is towards the mid-path and proportional to its distance from that point. When the body is at its mid-path, its acceleration is zero; hence there is no force acting upon it, and this position is one of equilibrium if the body has not any store of kinetic energy. Conversely, if a body has an acceleration proportional to its distance from a fixed point, O, it will have a simple harmonic motion. If the acceleration at unit distance from O is μ feet per second per second (corresponding to ω^2 in the case just considered), by describing a circle with centre O about its path as diameter, we can easily show that the body has simple harmonic motion, and by taking $\omega = \sqrt{\mu}$, μ corresponding to ω^2 in the above case, we can state its velocity and acceleration at a distance x from its centre of motion O, and its period of vibration, viz. velocity v at x feet from O is $\sqrt{\mu} . \sqrt{a^2 - x^2}$, or $\sqrt{\mu(a^2 - x^2)}$. Acceleration at x feet from centre O is $\mu . x$, and the time of a complete vibration is $\frac{2\pi}{\sqrt{\mu}}$.

Alternating Vectors.—We have seen that, the displacement of P being OP, the acceleration is proportional also to OP, and the velocity to OM; so that OP and OM are *vectors* representing in magnitude and direction the displacement and velocity of P. Such vectors, having a fixed end, O, and of length varying according to the position of a rotating vector, OQ or OS, are called "alternating vectors." It may be noted that the rate of change of an alternating vector, OP, of amplitude a is represented by another alternating vector, OM, of the same period, which is the projection of a uniformly rotating vector of length OS $= \omega . $OQ or ωa (to a different scale), and one right angle in advance of the rotating vector OQ, of which

OP is the projection. A little consideration will show that the rate of change of the alternating vector OM follows the same law (rate of change of velocity being acceleration), viz. it is represented by a third alternating vector, ON, of the same period, which is the projection of a uniformly rotating vector of length $OQ' = \omega . OS$ or $\omega^2 a$ (to a different scale), and one right angle in advance of the rotating vector OS, of which OM is the projection.

The curves of displacement, velocity, and acceleration of P on a base of angles are shown to the right hand of Fig. 50. The base representing angles must also represent time, since the rotating vectors have uniform angular velocity ω. The time $t = \frac{\theta}{\omega}$ seconds, since $\omega = \frac{\theta}{t}$. The properties of the curves of spaces, velocities, and accelerations (Arts. 4, 14, and 16) are well illustrated by the curves in Fig. 50, which have been drawn to three scales of space, velocity, and acceleration by projecting points 90° ahead of Q, S, and Q′ on the circle on the left. The acceleration of P, which is proportional to the displacement, may properly be considered to be of opposite sign to the displacement, since the acceleration is to the left from P to O when the displacement OP is to the right of O. The curves of displacement and acceleration are called "cosine curves," the ordinates being proportional to the cosines of angle $P\hat{O}Q$, or θ, or ωt. Similarly, the curve of velocity is called a "sine curve." The relations between the three quantities may be expressed thus—

Displacement (x) : velocity (v) : acceleration

$$= a \cos \omega t : a\omega \sin \omega t : -a\omega^2 \cos \omega t$$

Curved Path.—If the point P follows a curved path instead of the straight one AB, the curved path having the same length as the straight one, and if the acceleration of the point when distant x feet from its mid-path is tangential to the path and of the same magnitude as that of the point following the straight path AB when distant x feet from mid-path, then the velocity is of the same magnitude in each case. This is evident, for the points attain the same speeds in the

same intervals of time, being, under the same acceleration, always directed in the line of motion in each case. Hence the periodic times will be the same in each case, viz. $\frac{2\pi}{\sqrt{\mu}}$, where μ is the acceleration in feet per second per second along the curve or the straight line, as the case may be.

69. There are numerous instances in which bodies have simple harmonic motion or an approximation to it, for in perfectly elastic bodies the straining force is proportional to the amount of displacement produced, and most substances are very nearly perfectly elastic over a limited range.

A common case is that of a body hanging on a relatively light helical spring and vibrating vertically. The body is acted upon by an effective accelerating force proportional to its distance from its equilibrium position, and, since its mass does not change, it will have an acceleration $\left(\frac{\text{force}}{\text{mass}}\right)$ also proportional to its displacement from that point (Art. 40), and therefore it will vibrate with simple harmonic vibration.

Let W = weight of vibrating body in pounds.

e = force in pounds acting upon it at 1 foot from its equilibrium position, or per foot of displacement, the total displacement being perhaps less than 1 foot. This is sometimes called the *stiffness* of the spring.

Then $e.x$ = force in lbs. x feet from the equilibrium position

and if μ = acceleration in feet per second per second 1 foot from the equilibrium position or per foot of displacement

$$\mu = \frac{\text{accelerating force}}{\text{mass}} = e \div \frac{W}{g} = \frac{eg}{W}$$

hence the period of vibration is $\frac{2\pi}{\sqrt{\mu}}$ or $2\pi\sqrt{\frac{W}{eg}}$ (Art. 68)

The maximum force, which occurs when the extremities of the path are reached, is $e.a$, where a is the amplitude of

the vibration or distance from equilibrium position to either extremity of path, in feet.

The crank-pin of a steam engine describes a circle ABC (Fig. 51), of which the length of crank OC is the radius, with

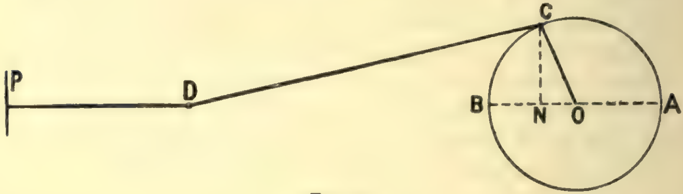


FIG. 51.

fairly constant angular velocity. The piston P and other reciprocating parts are attached to the crank-pin by a connecting-rod, DC, and usually move to and fro in a straight line, AP, with a diameter, AB, of the crank-pin circle. If the connecting-rod is very long compared to the crank-length, the motion is nearly the same as that of the projection N of the crank-pin on the diameter AB of the crank-pin circle, which is simple harmonic. If the connecting-rod is short, however, its greater obliquity modifies the piston-motion to a greater extent.

70. Energy stored in Simple Harmonic Motion.—If $e =$ force in pounds at unit distance, acting on a body of weight W lbs. having simple harmonic motion, the force at a distance x is ex, since it is proportional to the displacement. Therefore the work done in displacing the body from its equilibrium through x feet is $\frac{1}{2}ex^2$ (Art. 54 and Fig. 35). This energy, which is stored in some form other than kinetic energy when the body is displaced from its equilibrium position, reaches a maximum $\frac{1}{2}ea^2$ when the extreme displacement a (the amplitude) has taken place, and the effective accelerating force acting on the body is ea. In the mid-position of the body ($x = 0$), when its velocity is greatest and the force acting on it is nil, the energy is wholly kinetic, and in other intermediate positions the energy is partly kinetic and partly otherwise, the total being constant if there are no resistances.

Fig. 52 shows a diagram of work stored for various displacements of a body having simple harmonic motion. The amplitude OA $= a$, and therefore the force at A is ae, which is represented by AD, and the work done in moving from O to A is represented by the area AOD (Art. 54 and Fig. 35). At P, distant x feet from O, the work done in motion from O is $\frac{1}{2}ex^2$, represented by the area OHP, and the kinetic energy at P is therefore represented by the area DAPH.

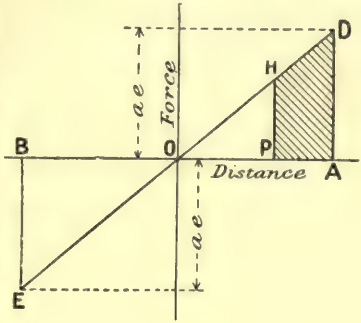


Fig. 52.

71. Simple Pendulum.—This name refers strictly to a particle of indefinitely small dimensions and yet having weight, suspended by a perfectly flexible weightless thread from a fixed point, about which, as a centre, it swings freely in a circular arc. In practice, a small piece of heavy metal, usually called a pendulum bob, suspended by a moderately long thin fibre, behaves very nearly indeed like the ideal pendulum defined above, the resistances, such as that of the atmosphere, being small.

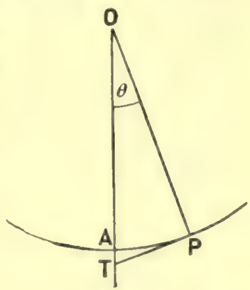


Fig. 53.

Let O, Fig. 53, be the point of suspension of the particle P of a simple pendulum.

Let OP, the length of thread, be l feet.

Let $\theta =$ angle $A\hat{O}P$ in radians which OP makes with the vertical (OA) through O in any position P of the particle.

Draw PT perpendicular to OP, *i.e.* tangent to the arc of motion to meet the vertical through O in T.

The tension of the thread has no component along the direction of motion (PT) at P. The acceleration along PT is

then $g \sin \theta$, since PT is inclined θ to the horizontal (Art. 28). If θ is very small, $\sin \theta$ may be taken equal to θ in radians. (If θ does not exceed 5°, the greatest error in this approximation is less than 1 part in 800.) Hence the acceleration along PT is $g\theta$ approximately. And $\theta = \frac{\text{arc AP}}{\text{radius OP}}$; therefore acceleration along PT $= \frac{g \times \text{arc AP}}{l}$, and the acceleration is proportional to the distance AP, along the arc, of P from A, being $\frac{g}{l}$ per foot of arc. Hence the time of a complete oscillation in seconds is—

$$2\pi \div \sqrt{\frac{g}{l}} = 2\pi\sqrt{\frac{l}{g}} \text{ (Art. 68)}$$

and the velocity at any point may be found, as in Art. 68, for any position of the swinging particle.

In an actual pendulum the pendulum bob has finite dimensions, and the length l will generally be somewhat greater than that of the fibre by which it is suspended. The ideal simple pendulum having the same period of swing as an actual pendulum of any form is called its *simple equivalent pendulum*. For this ideal pendulum the relation $t = 2\pi\sqrt{\frac{l}{g}}$ holds, and therefore $l = \frac{t^2 g}{4\pi^2}$, from which its length in feet may be calculated for a given time, t, of vibration.

The value of the acceleration of gravity, g, varies at different parts of the earth's surface, and the pendulum offers a direct means of measuring the value of this quantity g, viz. by accurate timing of the period of swing of a pendulum of known length. The length of an actual pendulum, *i.e.* of its simple equivalent pendulum, can be calculated from its dimensions.

Example 1.—A weight rests freely on a scale-pan of a spring balance, which is given a vertical simple harmonic vibration of period 0·5 second. What is the greatest amplitude the vibration may have in order that the weight may not leave the pan? What is then the pressure of the weight on the pan in its lowest position?

Let a = greatest amplitude in feet.

The greatest downward force on the body is its own weight, and therefore its greatest downward acceleration is g, occurring when the weight is in its highest position and the spring is about to return. Hence, if the scale-pan and weight do not separate, the downward acceleration of the pan must not exceed g, and therefore the acceleration must not exceed $\frac{g}{a}$ per foot of displacement.

The acceleration per foot of displacement is $\left(\frac{2\pi}{t}\right)^2$;

$$\text{therefore } \left(\frac{2\pi}{0{\cdot}5}\right)^2 \not> \frac{g}{a}$$

$$16\pi^2 \not> \frac{g}{a}$$

$$\text{or } a \not> \frac{32{\cdot}2}{16\pi^2} \text{ feet}$$

$$\textit{i.e. } a \not> 0{\cdot}204 \text{ feet or } 2{\cdot}448 \text{ inches}$$

If the balance has this amplitude of vibration, the pressure between the pan and weight at the lowest position will be equal to twice the weight, since there is an acceleration g upwards which must be caused by an effective force equal to the weight acting upwards, or a gross pressure of twice the weight from which the downward gravitational force has to be subtracted.

Example 2.—Part of a machine has a reciprocating motion, which is simple harmonic in character, making 200 complete oscillations in a minute; it weighs 10 lbs. Find (1) the accelerating force upon it in pounds and its velocity in feet per second, when it is 3 inches from mid-stroke; (2) the maximum accelerating force; and (3) the maximum velocity if its total stroke is 9 inches, *i.e.* if its amplitude of vibration is $4\frac{1}{2}$ inches.

$$\text{Time of 1 oscillation} = \frac{60}{200} = 0{\cdot}3 \text{ second}$$

$$\left.\begin{array}{l}\text{therefore the acceleration per foot}\\ \quad\text{distance from mid-stroke}\end{array}\right\} = \left(\frac{2\,\pi}{0{\cdot}3}\right)^2 = \frac{400\pi^2}{9} \text{ feet per second per second}$$

and the accelerating force 0·25 foot from mid-stroke on 10 lbs. is—

$$\frac{10}{32{\cdot}2} \times 0{\cdot}25 \times \frac{400\pi^2}{9} = 34{\cdot}08 \text{ lbs.}$$

and the maximum accelerating force $4\frac{1}{2}$ inches from mid-stroke is 1·5 times as much as at 3 inches, or $34{\cdot}08 \times 1{\cdot}5 = 51{\cdot}12$ lbs.

The maximum velocity in feet per second occurring at mid-stroke

$= \text{amplitude in feet} \times \sqrt{\text{acceleration per foot of displacement}}$ (Art. 68)

$= \text{amplitude in feet} \times \dfrac{2\pi}{\text{period}}$

$= \dfrac{3}{8} \times \dfrac{2\pi}{0{\cdot}3} = \dfrac{5\pi}{2} = 7{\cdot}85$ feet per second

Velocity at 3 inches from mid-stroke $\Big\} = 7{\cdot}85 \times \dfrac{\sqrt{4{\cdot}5^2 - 3^2}}{4{\cdot}5}$ (Art. 68)

$= 7{\cdot}85 \times \dfrac{\sqrt{11{\cdot}25}}{4{\cdot}5} = 5{\cdot}85$ feet per second

Example 3.—The crank of an engine makes 150 revolutions per minute, and is 1·3 feet long. It is driven by a piston and a very long connecting rod (Fig. 51), so that the motion of the piston may be taken as simple harmonic. Find the piston velocity and the force necessary to accelerate the piston and reciprocating parts, weighing altogether 300 lbs., (1) when the crank has turned through 45° from its position (OB) in line with and nearest to the piston path; (2) when the piston has moved forward 0·65 foot from the end of its stroke.

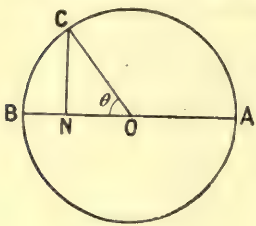


FIG. 54.

Let ABC (Fig. 54) be the circular path 1·3 feet radius of the crank-pin, CN the perpendicular from a point C on the diameter AB.

The angular velocity of crank OC is $\dfrac{150 \times 2\pi}{60} = 5\pi$ radians per second

(1) The motion of the piston being taken as that of N, the acceleration of piston when the crank-pin is at C is—

$$(5\pi)^2 \times 1{\cdot}3 \times \cos 45^\circ \qquad (\omega^2 r \cos\theta, \text{ Art. } 68)$$

and the accelerating force is—

$$\frac{300}{32{\cdot}2} \times (5\pi)^2 \times 1{\cdot}3 \times \frac{1}{\sqrt{2}} = 2110 \text{ lbs.}$$

The velocity is—

$$5\pi \times 1{\cdot}3 \times \sin 45^\circ = 14{\cdot}43 \text{ feet per second}$$

(2) When BN = 0·65 foot, ON = OB − BN = 1·3 − 0·65 = 0·65 foot, and $\hat{CON} = \cos^{-1}\dfrac{ON}{OC} = \cos^{-1}\frac{1}{2} = 60°$. The accelerating force is then—

$$\frac{300}{32·2} \times (5\pi)^2 \times 1·3 \times \tfrac{1}{2} = 1493 \text{ lbs.}$$

and the velocity is—

$$5\pi \times 1·3 \times \sin 60° = 17·67 \text{ feet per second}$$

Example 4.—A light helical spring is found to deflect 0·4 inch when an axial load of 4 lbs. is hung on it. How many vibrations per minute will this spring make when carrying a weight of 15 lbs.?

The force per foot of deflection is $4 \div \dfrac{0·4}{12} = 120$ lbs.

hence the time of vibration is $2\pi\sqrt{\dfrac{15}{32·2 \times 120}} = 0·392$ second

and the number of vibrations per minute is $\dfrac{60}{0·392} = 153·2$

Example 5.—Find the length of a clock pendulum which will make three beats per second. If the clock loses 1 second per hour, what change is required in the length of pendulum?

Let l = length of pendulum in feet.

Time of vibration = $\frac{1}{3}$ second

$$l = \frac{(\frac{1}{3})^2 \times 32·2}{4\pi^2} = \frac{32·2}{36\pi^2} \text{ feet} = 1·09 \text{ inches}$$

The clock loses 1 second in 3600 seconds, *i.e.* it makes 3599 × 3 beats instead of 3600 × 3. Since $l \propto t^2 \propto \dfrac{1}{n^2}$, where n = number of beats per hour, therefore—

$$\frac{\text{correct length}}{1·09 \text{ inches}} = \frac{3599^2}{3600^2} = (1 - \tfrac{1}{3600})^2$$
$$= 1 - \tfrac{1}{1800} \text{ approximately}$$

therefore *shortening* required = $\dfrac{1·09}{1800}$ inches = 0·000606 inch

Examples IX.

1. A point has a simple harmonic motion of amplitude 6 inches and period 1·5 seconds. Find its velocities and accelerations 0·1 second, 0·2 second, and 0·5 second after it has left one extremity of its path.

2. A weight of 10 lbs. hangs on a spring, which stretches 0·15 inch per pound of load. It is set in vibration, and its greatest acceleration whilst in motion is 16·1 feet per second per second. What is the amplitude of vibration?

3. A point, A, in a machine describes a vertical circle of 3 feet diameter, making 90 rotations per minute. A portion of the machine weighing 400 lbs. moves in a horizontal straight line, and is always a fixed distance horizontally from A, so that it has a stroke of 3 feet. Find the accelerating forces on this portion, (1) at the end of its stroke; (2) 9 inches from the end; and (3) 0·05 second after it has left the end of its stroke.

4. A helical spring deflects $\frac{1}{8}$ of an inch per pound of load. How many vibrations per minute will it make if set in oscillation when carrying a load of 12 lbs.?

5. A weight of 20 lbs. has a simple harmonic vibration, the period of which is 2 seconds and the amplitude 1·5 feet. Draw diagrams to stated scales showing (1) the net force acting on the weight at all points in its path; (2) the displacement at all *times* during the period; (3) the velocity at all times during the period; (4) the force acting at all times during the period.

6. A light stiff beam deflects 1·145 inches under a load of 1 ton at the middle of the span. Find the period of vibration of the beam when so loaded.

7. A point moves with simple harmonic motion; when 0·75 foot from mid-path, its velocity is 11 feet per second; and when 2 feet from the centre of its path, its velocity is 3 feet per second. Find its period and its greatest acceleration.

8. How many complete oscillations per minute will be made by a pendulum 3 feet long? $g = 32·2$.

9. A pendulum makes 3000 beats per hour at the equator, and 3011 per hour near the pole. Compare the value of g at the two places.

CHAPTER V

STATICS—CONCURRENT FORCES—FRICTION

72. THE particular case of a body under the action of several forces having a resultant zero, so that the body remains at rest, is of very common occurrence, and is of sufficient importance to merit special consideration. The branch of mechanics which deals with bodies at rest is called Statics.

We shall first consider the statics of a particle, *i.e.* a body having weight, yet of indefinitely small dimensions. Many of the conclusions reached will be applicable to small bodies in which all the forces acting may be taken without serious error as acting at the same point, or, in other words, being concurrent forces.

73. Resolution and Composition of Forces in One Plane.—It will be necessary to recall some of the conclusions of Art. 44, viz. that any number of concurrent forces can be replaced by their geometric sum acting at the intersection of the lines of action of the forces, or by components in two standard directions, which are for convenience almost always taken at right angles to one another.

Triangle and Polygon of Forces.—If several forces, say four, as in Fig. 55, act on a particle, and *ab*, *bc*, *cd*, *de* be drawn in succession to represent the forces of 7, 8, 6, and 10 lbs. respectively, then *ae*, their geometric sum (Art. 44), represents a force which will produce exactly the same effect as the four forces, *i.e.* *ae* represents the resultant of the four forces. If the final point *e* of the polygon *abcde* coincides with the point *a*, then the resultant *ae* is nil, and the four forces are in equilibrium. This proposition is called the Polygon of Forces, and may be

stated as follows: If several forces acting on a particle be represented in magnitude and direction by the sides of a closed polygon taken in order, they are in equilibrium. By a closed polygon is meant one the last side of which ends at the point

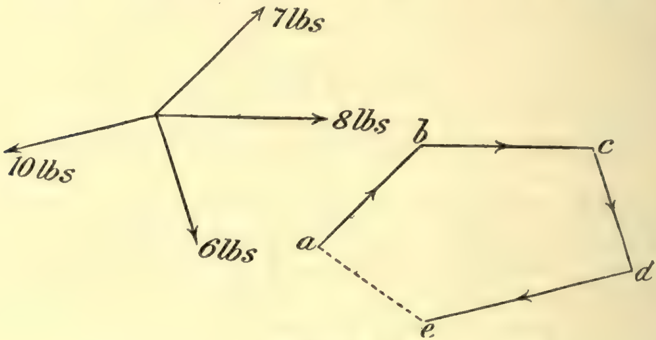


FIG. 55.

from which the first side started. The intersection of one side of the polygon with other sides is immaterial.

The polygon of forces may be proved experimentally by means of a few pieces of string and weights suspended over almost frictionless pulleys, or by a number of spring balances and cords.

This proposition enables us to find one force out of several keeping a body in equilibrium if the remainder are known, viz. by drawing to scale an open polygon of vectors corresponding to the known forces, and then a line joining its extremities is the vector representing in one direction the resultant of the other forces or in the other direction the remaining force necessary to maintain equilibrium, sometimes called the *equilibrant*.

For example, if forces Q, R, S, and T (Fig. 56) of given magnitudes, and one other force keep a particle P in equilibrium, we can find the remaining one as follows. Set out vectors *ab*, *bc*, *cd*, and *de* in succession to represent Q, R, S, and T respectively; then *ae* represents their resultant in magnitude and direction, and *ea* represents in magnitude and direction the remaining force which would keep the particle P in equilibrium, or the equilibrant.

Similarly, if all the forces keeping a body in equilibrium except two are known, and the directions of these two are known, their magnitudes may be found by completing the

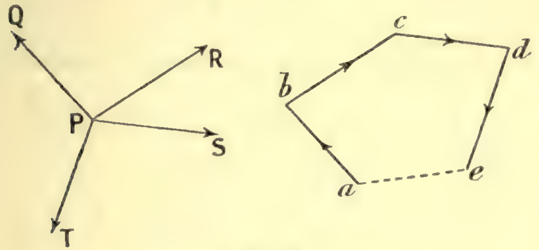


FIG. 56.

open vector polygon by two intersecting sides in the given directions.

In the particular case of three forces keeping a body in equilibrium, the polygon is a triangle, which is called the Triangle of Forces. Any triangle having its sides respectively parallel to three forces which keep a particle in equilibrium represents by its sides the respective forces, for a three-sided closed vector polygon (*i.e.* a triangle) with its sides parallel and proportional to the forces can always be drawn as directed for the polygon of forces, and any other triangle with its sides parallel to those of this vector triangle has its sides also proportional to them, since all triangles with sides respectively parallel are similar. The corresponding proposition as to *any* polygon with sides parallel to the respective forces is not true for any number of forces but three.

74. Lami's Theorem.—If three forces keep a particle in equilibrium, each is proportional to the sine of the angle between the other two.

Let P, Q, and R (Fig. 57) be the three forces in equilibrium acting at O in the lines OP, OQ, and OR respectively. Draw any three non-concurrent lines parallel respectively to OP, OQ, and OR, forming a triangle *abc* such that *ab* is parallel to OP, *bc* to OQ, and *ca* to OR. Then angle $a\hat{b}c = 180 - P\hat{O}Q$, angle

$\hat{bca} = 180 - \hat{QOR}$, and angle $\hat{cab} = 180 - \hat{ROP}$, and therefore—

$$\sin \hat{abc} = \sin \hat{POQ}$$
$$\sin \hat{bca} = \sin \hat{QOR}$$
$$\sin \hat{cab} = \sin \hat{ROP}$$

In the last article, it was shown that any triangle, such as

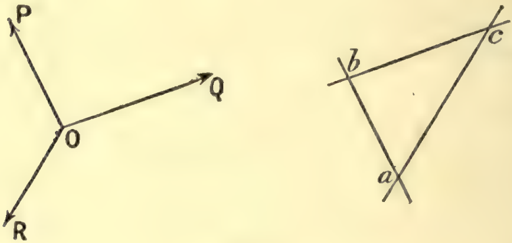


FIG. 57.

abc, having sides respectively parallel to OP, OQ, and OR, has its sides proportional respectively to P, Q, and R, or—

$$\frac{P}{ab} = \frac{Q}{bc} = \frac{R}{ca} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{also } \frac{ab}{\sin \hat{bca}} = \frac{bc}{\sin \hat{cab}} = \frac{ca}{\sin \hat{abc}}$$

$$\text{or } \frac{ab}{\sin \hat{QOR}} = \frac{bc}{\sin \hat{ROP}} = \frac{ca}{\sin \hat{POQ}} \quad . \quad . \quad (2)$$

and multiplying equation (1) by equation (2)—

$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}$$

that is, each of the forces P, Q, and R is proportional to the sine of the angle between the other two.

This result is sometimes of use in solving problems in which three forces are in equilibrium.

75. Analytical Methods.—Resultant or equilibrant forces of a system, being representable by vectors, may be found by the rules used for resultant velocities, *i.e.* (1) by drawing

vectors to scale; (2) by the rules of trigonometry for the solutions of triangles; (3) by resolution into components in two standard directions and subsequent compounding as in Art. 25. We now proceed to the second and third methods.

To compound two forces P and Q inclined at an angle θ to each other.

Referring to the vector diagram *abc* of Fig. 58 (which need

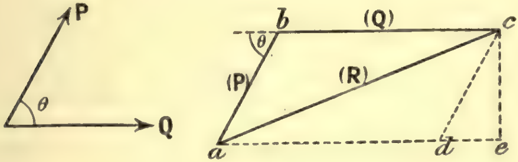


FIG. 58.

not be drawn, and is used here for the purpose of illustration and explanation) by the rules of trigonometry for the solution of triangles—

$$(ac)^2 = (ab)^2 + (bc)^2 - 2\ ab\,.\,bc \cos \hat{abc}$$
$$= (ab)^2 + (bc)^2 + 2\ ab\,.\,bc \cos \theta$$

hence if *ab* and *bc* represent P and Q respectively, and R is the value of their resultant—

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

from which R may be found by extracting the square root, and its inclination to, say, the direction of Q may be found by considering the length of the perpendicular *ce* from *c* on *ad* produced—

$$\text{Since } ec = dc \sin \theta$$
$$\text{and } de = dc \cos \theta$$
$$\tan \hat{cad} = \frac{ec}{ae} = \frac{dc \sin \theta}{ad + dc \cos \theta} = \frac{P \sin \theta}{Q + P \cos \theta}$$

which is the tangent of the angle between the line of action of the resultant R and that of the force Q.

When the resultant or equilibrant of more than two concurrent forces is to be found, the method of Art. 25 is

sometimes convenient. Suppose, say, three forces F_1, F_2, and F_3 make angles α, β, and γ respectively with some chosen fixed direction OX, say that of the line of action of F_1, so that $\alpha = 0$ (Fig. 59).

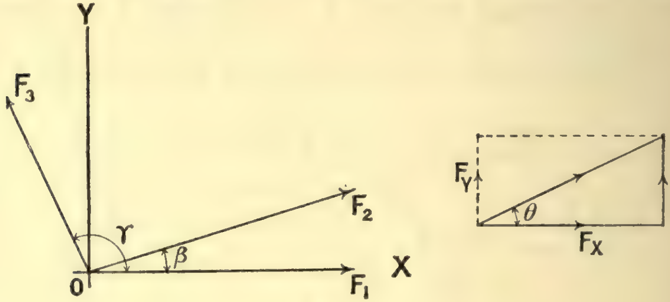


FIG. 59.

Resolve F_1, F_2, and F_3 along OX and along OY perpendicular to OX.

Let F_X be the total of the components along OX,
and let F_Y " " " " " OY.

Let R be the resultant force, and θ its inclination to OX; then—

$$F_X = F_1 + F_2 \cos \beta + F_3 \cos \gamma$$
$$F_Y = 0 + F_2 \sin \beta + F_3 \sin \gamma$$

and compounding F_X and F_Y, two forces at right angles, R is proportional to the hypotenuse of a right-angled triangle, the other sides of which are proportional to F_X and F_Y; hence—

$$R^2 = F_X{}^2 + F_Y{}^2$$
$$\text{and } R = \sqrt{(F_X{}^2 + F_Y{}^2)}$$

The direction of the resultant R is given by the relation—

$$\tan \theta = \frac{F_Y}{F_X}$$

If the forces of the system are in equilibrium, that is, if the resultant is nil—

$$R^2 = 0$$
$$\text{or } F_X{}^2 + F_Y{}^2 = 0$$

This is only possible if both $F_X = 0$ and $F_Y = 0$.

The condition of equilibrium, then, is, that the components in each of two directions at right angles shall be zero. This corresponds to the former statement, that if the forces are in equilibrium, the vector polygon of forces shall be closed, as will be seen by projecting on any two fixed directions at right angles, the sides of the closed polygon, taking account of the signs of the projections. The converse statement is true, for if $F_X = 0$ and $F_Y = 0$, then $R = 0$; therefore, if the components in each of two standard directions are zero, then the forces form a system in equilibrium, corresponding to the statement that if the vector polygon is a closed figure, the forces represented by its sides are in equilibrium.

Example 1.—A pole rests vertically with its base on the ground, and is held in position by five ropes, all in the same horizontal plane and drawn tight. From the pole the first rope runs due north, the second 75° west of north, the third 15° south of west, and the fourth 30° east of south. The tensions of these four are 25 lbs.,

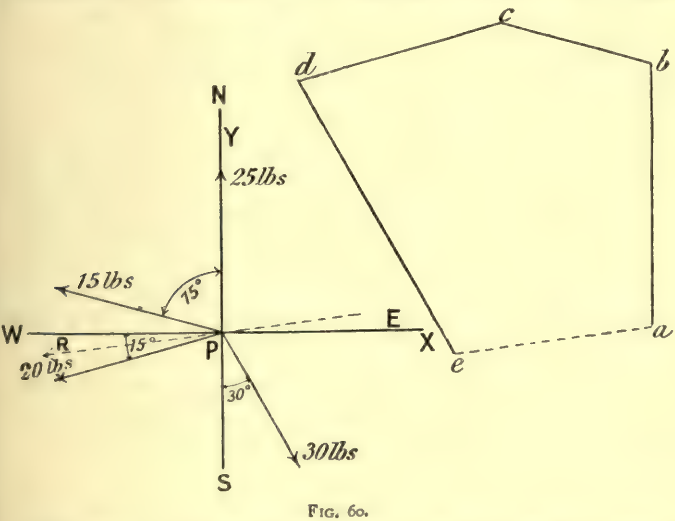


FIG. 60.

15 lbs., 20 lbs., and 30 lbs. respectively. Find the direction of the fifth rope and its tension.

The directions of the rope have been set out in Fig. 60, which

represents a plan of the arrangement, the pole being at P. The vector polygon *abcde*, representing the forces in the order given, has been set out from *a* and terminates at *e*. *ae* has been drawn, and measures to scale 18·9 lbs., and the equilibrant *ea* is the pull in the fifth rope, and its direction is 7° north of east from the pole.

Example 2.—Two forces of 3 lbs. and 5 lbs. respectively act on a particle, and their lines of action are inclined to each other at an angle of 70°. Find what third force will keep the particle in equilibrium.

The resultant force R will be of magnitude given by the relation—

$$R^2 = 3^2 + 5^2 + 2 . 3 . 5 \cos 70^\circ$$
$$= 9 + 25 + (30 \times 0·3420) = 34 + 10·26 = 44·26$$
$$R = \sqrt{44·42} = 6·65 \text{ lbs.}$$

And R is inclined to the force of 5 lbs. at an angle the tangent of which is—

$$\frac{3 \sin 70^\circ}{5 + 3 \cos 70^\circ} = \frac{3 \times 0·9397}{5 + (3 \times 0·3420)}$$
$$= \frac{2·8171}{6·026} = 0·467$$

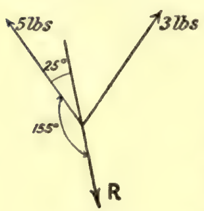


Fig. 61.

which is an angle 25°. The equilibrant or third force required to maintain equilibrium is, therefore, one of 6·65 lbs., and its line of action makes an angle of 180° − 25° or 155° with the line of action of the force of 5 lbs., as shown in Fig. 61.

Example 3.—Solve Example 1 by resolving the forces into components. Taking an axis PX due east (Fig. 60) and PY due north, component force along PX—

$$F_X = -15 \cos 15^\circ - 20 \cos 15^\circ + 30 \cos 60^\circ$$
$$= (-35 \times 0·9659) + (30 \times 0·5) = -18·806 \text{ lbs.}$$

Component force along PY—

$$F_Y = 25 + 15 \cos 75^\circ - 20 \cos 75^\circ - 30 \cos 30^\circ$$
$$= 25 - (5 \times 0·2588) - 30 \times 0·8660 = -2·274 \text{ lbs.}$$
$$\text{hence } R^2 = (18·81)^2 + (2·27)^2 = 359·3$$
$$R = \sqrt{359·3} = 18·96 \text{ lbs.}$$

R acts outwards from P in a direction south of west, being inclined to XP at an acute angle, the tangent of which is—

$$\frac{F_Y}{F_X} = \frac{2 \cdot 274}{18 \cdot 806} = 0 \cdot 121$$

which is the tangent of 6° 54′; *i.e.* R acts in a line lying 6° 54′ south of west. The equilibrant is exactly opposite to this, hence the fifth rope runs outwards from the pole P in a direction 6° 54′ north of east, and has a tension of 18·96 lbs.

Examples X.

1. A weight of 20 lbs. is supported by two strings inclined 30° and 45° respectively to the horizontal. Find by graphical construction the tension in each cord.

2. A small ring is situated at the centre of a hexagon, and is supported by six strings drawn tight, all in the same plane and radiating from the centre of the ring, and each fastened to a different angular point of the hexagon. The tensions in four consecutive strings are 2, 7, 9, and 6 lbs. respectively. Find the tension in the two remaining strings.

3. Five bars of a steel roof-frame, all in one plane, meet at a point; one is a horizontal tie-bar carrying a tension of 40 tons; the next is also a tie-bar inclined 60° to the horizontal and sustaining a pull of 30 tons; the next (in continuous order) is vertical, and runs upward from the joint, and carries a thrust of 5 tons; and the remaining two in the same order radiate at angles of 135° and 210° to the first bar. Find the stresses in the last two bars, and state whether they are in tension or compression, *i.e.* whether they pull or push at the common joint.

4. A telegraph pole assumed to have no force bending it out of the vertical has four sets of horizontal wires radiating from it, viz. one due east, one north-east, one 30° north of west, and one other. The tensions of the first three sets amount to 400 lbs., 500 lbs., and 250 lbs. respectively. Find, by resolving the forces north and east, the direction of the fourth set and the total tension in it.

5. A wheel has five equally spaced radial spokes, all in tension. If the tensions of three consecutive spokes are 2000 lbs., 2800 lbs., and 2400 lbs. respectively, find the tensions in the other two.

6. Three ropes, all in the same vertical plane, meet at a point, and there support a block of stone. They are inclined at angles of 40°, 120°, and 160° to a horizontal line in their common plane. The pulls in the first two ropes are 150 lbs. and 120 lbs. respectively. Find the weight of the block of stone and the tension in the third rope.

76. Friction.—Friction is the name given to that property of two bodies in contact, by virtue of which a resistance

is offered to any sliding motion between them. The resistance consists of a force tangential to the surface of each body at the place of contact, and it acts on each body in such a direction as to oppose relative motion. As many bodies in equilibrium are held in their positions partly by frictional forces, it will be convenient to consider here some of the laws of friction.

77. The laws governing the friction of bodies at rest are found by experiment to be as follows :—

(1) *The force of friction always acts in the direction opposite to that in which motion would take place if it were absent, and adjusts itself to the amount necessary to maintain equilibrium.*

There is, however, a limit to this adjustment and to the value which the frictional force can reach in any given case. This maximum value of the force of friction is called the limiting friction. It follows the second law, viz.—

(2) *The limiting friction for a given pair of surfaces depends upon the nature of the surfaces, is proportional to the normal pressure between them, and independent of the area of the surfaces in contact.*

For a pair of surfaces of a given kind (*i.e.* particular substances in a particular condition), the limiting friction $F = \mu . R$, where R is the normal pressure between the surfaces, and μ is a constant called the *coefficient of friction* for the given surfaces. This second law, which is deduced from experiment, must be taken as only holding approximately.

78. Friction during Sliding Motion.—If the limiting friction between the bodies is too small to prevent motion, and sliding motion begins, the subsequent value of the frictional force is somewhat less than that of the statical friction. The laws of friction of motion, so far as they have been exactly investigated, are not simple. The friction is affected by other matter (such as air), which inevitably gets between the two surfaces. However, for very low velocities of sliding and moderate normal pressure, the same relations hold approximately as have been stated for the limiting friction of rest, viz.—

$$F = \mu R$$

where F is the frictional force between the two bodies, and R

is the normal pressure between them, and μ is a constant coefficient for a given pair of surfaces, and which is less than that for statical friction between the two bodies. The friction is also independent of the velocity of rubbing.

79. Angle of Friction.—Suppose a body A (Fig. 62) is in contact with a body B, and is being pulled, say, to the right, the pull increasing until the limiting amount of frictional resistance is reached, that is, until the force of friction reaches a limiting value $F = \mu R$, where R is the normal pressure between

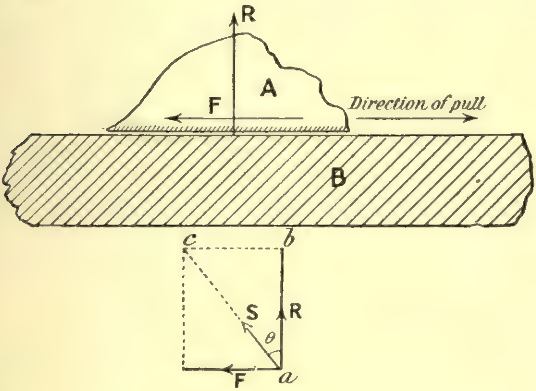


FIG 62.

the two bodies, and μ is the coefficient of friction. If R and F, which are at right angles, are compounded, we get the resultant pressure, S, which B exerts on A. As the friction F increases with the pull, the inclination θ of the resultant S of F and R to the normal of the surface of contact, *i.e.* to the line of action of R, will become greater, since its tangent is always equal to $\frac{cb}{ab}$ or $\frac{F}{R}$ (Art. 75).

Let the extreme inclination to the normal be λ when the friction F has reached its limit, μR.

$$\tan \lambda = \frac{F}{R} = \frac{\mu R}{R} = \mu$$

This extreme inclination, λ, of the resultant force between

two bodies to the normal of the common surface in contact is called the *angle of friction*, and we have seen that it is the angle the tangent of which is equal to the coefficient of friction—

$$\tan \lambda = \mu, \text{ or } \lambda = \tan^{-1} \mu$$

80. Equilibrium of a Body on an Inclined Plane.—As a simple example of a frictional force, it will be instructive here to consider the equilibrium of a body resting on an inclined plane, supported wholly or in part by the friction between it and the inclined plane.

Let μ be the coefficient of friction between the body of weight W and the inclined plane, and let α be the inclination of the plane to the horizontal plane. We shall in all cases draw the vector polygon of forces maintaining equilibrium, not necessarily correctly to scale, and deduce relations between the forces by the trigonometrical relations between the parts of the polygon, thus combining the advantages of vector illustration with algebraic calculation, as in Art. 75. The normal to the plane is shown dotted in each diagram (Figs. 63–68 inclusive).

1. Body at rest on an inclined plane (Fig. 63).

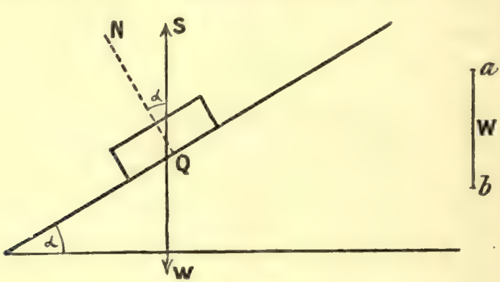


FIG. 63.

If the body remains at rest unaided, there are only two forces acting on it, viz. its weight, W, and the reaction S of the plane; these must then be in a straight line, and therefore S must be vertical, *i.e.* inclined at an angle α to the normal to the plane. The greatest angle which S can make to the normal

is λ, the angle of friction (Art. 79); therefore α cannot exceed λ, the angle of friction, or the body would slide down the plane. Thus we might also define the angle of friction between a pair of bodies as the greatest incline on which one body would remain on the other without sliding.

Proceeding to supported bodies, let an external force, P, which we will call the effort, act upon the body in stated directions.

2. Horizontal effort necessary to start the body up the plane. Fig. 64 shows the forces acting, and a triangle of forces, *abc*.

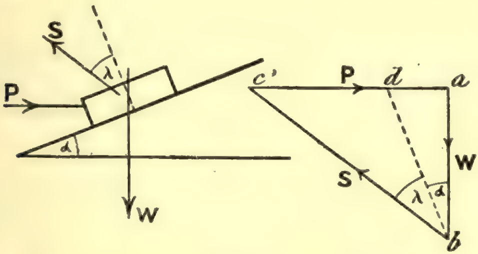


FIG. 64.

When the limit of equilibrium is reached, and the body is about to slide up the plane, the angle *dbc* will be equal to λ, the maximum angle which S can make with the normal to the plane; then—

$$\frac{P}{W} = \frac{ca}{ab} = \tan(\alpha + \lambda)$$
$$\text{or } P = W \tan(\alpha + \lambda)$$

which is the horizontal effort necessary to start the body up the plane.

3. Horizontal effort necessary to start the body sliding down the plane (Fig. 65).

When the body is about to move down the plane, the angle $\hat{cbd}$ will be equal to the angle of friction, λ; then—

$$\frac{P}{W} = \frac{ca}{ab} = \tan(\lambda - \alpha)$$
$$\text{or } P = W \tan(\lambda - \alpha)$$

If α is greater than λ, this can only be negative, *i.e.* c falls to the left of a, and the horizontal force P is that necessary

$ab = \mathrm{W}$
$bc = \mathrm{S}$
$ca = \mathrm{P}$

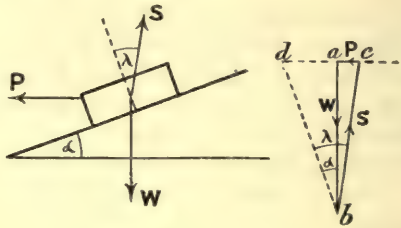


FIG. 65.

to just support the body on the steep incline on which it cannot rest unsupported.

4. Effort required parallel to the plane to start the body up the plane (Fig. 66).

angle $a\hat{c}b = 90^\circ - \lambda$
$ab = \mathrm{W}$
$bc = \mathrm{S}$
$ca = \mathrm{P}$

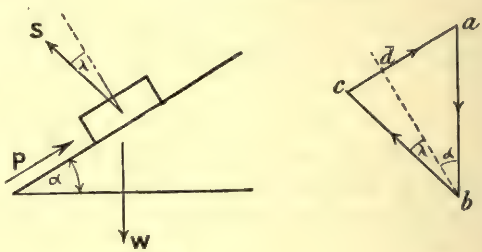


FIG. 66.

When the body is about to slide up the plane, the reaction S will make its maximum angle λ ($d\hat{b}c$) to the normal.

$$\text{Then } \frac{\mathrm{P}}{\mathrm{W}} = \frac{ac}{ab} = \frac{\sin(\lambda + \alpha)}{\sin(90^\circ - \lambda)}$$

$$\text{or } \mathrm{P} = \mathrm{W}\frac{\sin(\lambda + \alpha)}{\cos \lambda}$$

which is the effort parallel to the plane necessary to start the body moving up the plane.

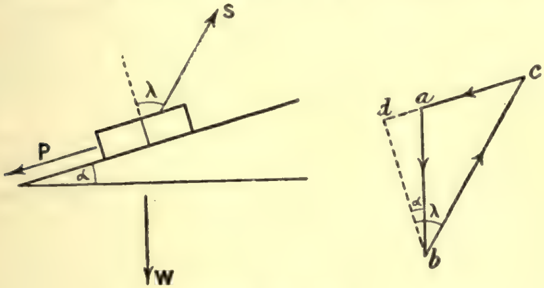
5. Effort required parallel to the plane to start the body down the plane (Fig. 67).

When the body is just about to slide down the plane, $\hat{cbd} = \lambda$.

$$\text{Then } \frac{P}{W} = \frac{ca}{ab} = \frac{\sin(\lambda - \alpha)}{\sin(90^\circ - \lambda)}$$

$$\text{or } P = W\frac{\sin(\lambda - \alpha)}{\cos \lambda}$$

which is the least force parallel to the plane necessary to start the body moving down the plane. If α is greater than λ, this



angle $a\hat{c}b = 90^\circ - \lambda$
$ab = W$
$bc = S$
$ca = P$

FIG. 67.

force, P, can only be negative, i.e. c falls between a and d, and the force is then that parallel to the plane necessary to just support the body from sliding down the steep incline.

6. Least force necessary to start the body up the incline.

Draw ab (Fig. 68) to represent W, and a vector, bc, of indefinite length to represent S inclined λ to the normal. Then the vector joining a to the line bc is least when it is perpendicular to bc. Then P is least when its line of action is perpendicular to that of S; that is, when it is inclined $90^\circ - \lambda$ to the normal, or λ to the plane; and then—

$$\frac{P}{W} = \frac{ca}{ab} = \sin(\alpha + \lambda)$$

Note that when $\alpha = 0$,

$$P = W \sin \lambda$$

which is the least force required to draw a body along the level.

$ab = W$
$bc = S$
$ca = P$

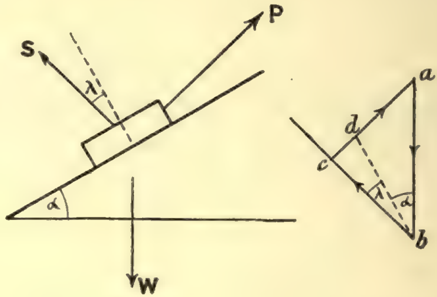


Fig. 68.

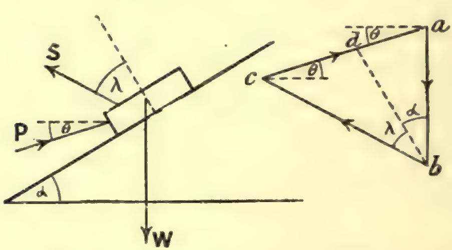
7. Similarly, the least force necessary to start the body down a plane inclined α to the horizontal is—

$$P = W \sin (\lambda - \alpha)$$

if λ is greater than α. If α is greater than λ, P is negative, and P is the least force which will support the body on the steep incline. In either case, P is inclined $90° - \lambda$ to the normal or λ to the plane.

8. Effort required in any assigned direction to start the body up the plane.

Let θ be the assigned angle which the effort P makes with the horizontal (Fig. 69).



angle $b\hat{a}c = 90° - \theta$
$a\hat{c}b = 90° - \alpha - \lambda + \theta$

$ab = W$
$bc = S$
$ca = P$

Fig. 69.

$$\text{Then } \frac{P}{W} = \frac{ca}{ab} = \frac{\sin(\lambda + \alpha)}{\sin acb} = \frac{\sin(\lambda + \alpha)}{\cos\{\theta - (\alpha + \lambda)\}}$$

$$\text{or } P = W\frac{\sin(\lambda + \alpha)}{\cos\{\theta - (\lambda + \alpha)\}}$$

which is the effort necessary to start the body *up* the plane in the given direction.

9. The effort in any assigned direction necessary to pull the body down the plane may be similarly found, the resultant force S between the body and plane acting in this case at an angle λ to the normal, but on the opposite side from that on which it acts in case 8.

81. Action of Brake-blocks: Adhesion.—A machine or vehicle is often brought to rest by opposing its motion by a frictional force at or near the circumference of a wheel or a drum attached to the wheel. A block is pressed against the rotating surface, and the frictional force tangential to the direction of rotation does work in opposing the motion. The amount of work done at the brake is equal to the diminution of kinetic energy, and this fact gives a convenient method of making calculations on the retarding force. The force is not generally confined to what would usually be called friction, as frequently considerable abrasion of the surface takes place, and the blocks wear away. It is usual to make the block of a material which will wear more rapidly than the wheel or drum on which it rubs, as it is much more easily renewed. If the brake is pressed with sufficient force, or the coefficient of "brake friction" between the block and the wheel is sufficiently high, the wheel of a vehicle may cease to rotate, and begin to slide or skid along the track. This limits the useful retarding force of a brake to that of the sliding friction between the wheels to which the brake is applied and the track, a quantity which may be increased by increasing the proportion of weight on the wheels to which brakes are applied. The coefficient of sliding friction between the wheels and the track is sometimes called the *adhesion*, or *coefficient of adhesion.*

82. Work spent in Friction.—If the motion of a body is opposed by a frictional force, the amount of work done

against friction in foot-pounds is equal to the force in pounds tangential to the direction of motion, multiplied by the distance in feet through which the body moves at the point of application of the force.

If the frictional force is applied at the circumference of a cylinder, as in the case of a brake band or that of a shaft or journal revolving in a bearing, the force is not all in the same line of action, but is everywhere tangential to the rotating cylinder, and it is convenient to add the forces together arithmetically and consider them as one force acting tangentially to the cylinder in any position, opposing its motion. If the cylinder makes N rotations per minute, and is R feet radius, and the tangential frictional force at the circumference of the cylinder is F lbs., then the work done in one rotation is $2\pi R . F$ foot-lbs., and the work done per minute is $2\pi RF . N$ foot-lbs., and the power absorbed is $\frac{2\pi R . F . N}{33,000}$ horse-power (Art. 55).

In the case of a cylindrical journal bearing carrying a resultant load W lbs., $F = \mu W$, where μ is the coefficient of friction between the cylinder and its bearing.

83. Friction and Efficiency of a Screw.—The screw is a simple application of the inclined plane, the thread on

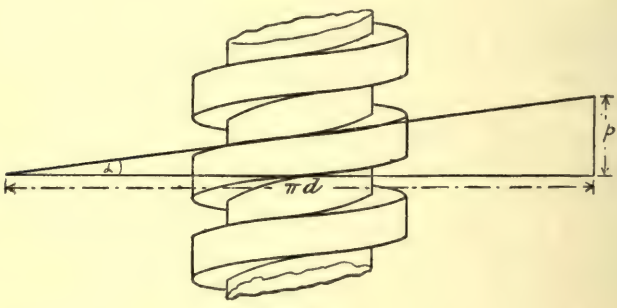


FIG. 70.

either the screw or its socket (or nut) fulfilling the same functions as a plane of the same slope. For simplicity a square-threaded screw (Fig. 70) in a vertical position is considered, the diameter

d inches being reckoned as twice the mean distance of the thread from the axis.

Let p = the pitch or axial distance, say in inches, from any point on the thread to the next corresponding point, so that when the screw is turned through one complete rotation in its fixed socket it rises p inches. Then the tangent of the angle of slope of the screw thread at its mean distance is $\frac{p}{\pi d}$, which corresponds to tan α in Art. 80. Hence, if a tangential horizontal effort P lbs. be applied to the screw at its mean diameter in order to raise a weight W lbs. resting on the top of the screw—

$$\frac{P}{W} = \tan (\alpha + \lambda)$$

where $\tan \lambda = \mu$ (Art. 80 (2)); or, expanding $\tan (\alpha + \lambda)$—

$$\frac{P}{W} = \frac{\tan \alpha + \tan \lambda}{1 - \tan \alpha \tan \lambda} = \frac{\frac{p}{\pi d} + \mu}{1 - \frac{\mu p}{\pi d}} = \frac{p + \mu \pi d}{\pi d - \mu p}$$

which has the value $\frac{p}{\pi d}$ or tan α for a frictionless screw.

Again, the work spent per turn of the screw is—

$$P \times \pi d = W(\tan \alpha + \lambda) \,.\, \pi d \text{ inch-lbs.}$$

The useful work done is $W \,.\, p$ inch-lbs.; therefore the work lost in friction is $W \tan (\alpha + \lambda)\pi d - Wp$ foot-lbs., an expression which may be put in various forms by expansion and substitution. The "efficiency" or proportion of useful work done to the total expenditure of work is—

$$\frac{Wp}{W \tan (\alpha + \lambda)\pi d} = \frac{\tan \alpha}{\tan (\alpha + \lambda)}$$

which may also be expressed in terms of p, d, and μ. The quantity $\frac{W}{P}$ is called the mechanical advantage; it is the ratio of the load to the effort exerted, and is a function of the

dimensions and the friction which usually differs with different loads.

84. Friction of Machines.—Friction is exerted at all parts of a machine at which there is relative tangential motion of the parts. It is found by experiment that its total effects are such that the relation between the load and the effort, between the load and the friction, and between the load and the efficiency generally follow remarkably simple laws between reasonable limits. The subject is too complex for wholly theoretical treatment, and is best treated experimentally. It is an important branch of practical mechanics.

Example 1.—A block of wood weighing 12 lbs. is just pulled along over a horizontal iron track by a horizontal force of $3\frac{1}{2}$ lbs. Find the coefficient of friction between the wood and the iron. How much force would be required to drag the block horizontally if the force be inclined upwards at an angle of 30° to the horizontal?

If μ = the coefficient of friction—

$$\mu \times 12 = 3\tfrac{1}{2} \text{ lbs.}$$

$$\mu = \frac{3 \cdot 5}{12} = 0 \cdot 291$$

Let P = force required at 30° inclination;

S = resultant force between the block and the iron track.

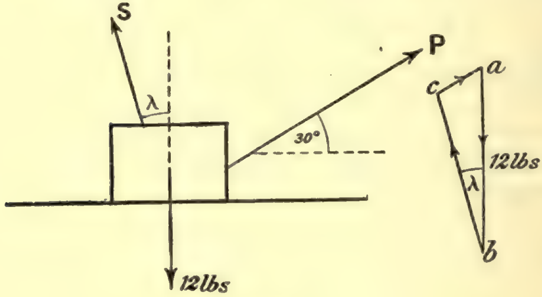


FIG. 71.

abc (Fig. 71) shows the triangle of forces when the block just reaches limiting equilibrium. In this triangle, $\hat{cab}$ = 60°, since P is inclined 30° to the horizontal; and—

$$\tan \hat{abc} = \mu = 0{\cdot}291 \text{ or } \tfrac{7}{24}$$

$$\text{hence } \sin abc = \frac{1}{\sqrt{(1 + \cot^2 \hat{abc})}} = \frac{1}{\sqrt{\{1 + (\frac{24}{7})^2\}}} = \frac{7}{25} = \sin \lambda$$

$$\text{and } \cos \lambda = \tfrac{24}{25}$$

$$\frac{P}{12} = \frac{ca}{ab} = \frac{\sin \hat{abc}}{\sin \hat{acb}} = \frac{\sin \lambda}{\sin (\lambda + 60)} = \frac{\sin \lambda}{\frac{1}{2} \sin \lambda + \frac{\sqrt{3}}{2} \cos \lambda}$$

$$= \frac{7 \times 2}{7 + 24\sqrt{3}} = 0{\cdot}289$$

$$P = 12 \times 0{\cdot}289 = 3{\cdot}46 \text{ lbs.}$$

Or thus—

$$\left.\begin{array}{l}\text{Normal pressure between block}\\ \quad\text{and track}\end{array}\right\} = 12 - P \sin 30^\circ$$

$$\text{horizontal pull } P \cos 30^\circ = \mu(12 - P \sin 30^\circ)$$

$$P\left(\frac{\sqrt{3}}{2} + \frac{7}{48}\right) = 12 \times \tfrac{7}{24}$$

$$\text{hence } P = 3{\cdot}46 \text{ lbs.}$$

Example 2.—A train, the weight of which, including locomotive, is 120 tons, is required to accelerate to 40 miles per hour from rest in 50 seconds. If the coefficient of adhesion is $\frac{1}{7}$, find the necessary weight on the driving wheels. In what time could the train be brought to rest from this speed, (1) with continuous brakes (*i.e.* on every wheel on the train); (2) with brakes on the driving-wheels only?

The acceleration is $\frac{2}{3} \times 88 \times \frac{1}{50} = 1{\cdot}17\dot{3}$ feet per sec. per sec.
The accelerating force is $1{\cdot}17\dot{3} \times \dfrac{120}{32{\cdot}2} = 4{\cdot}37$ tons

The greatest accelerating force obtainable without causing the driving-wheels to slip is $\frac{1}{7}$ of the weight on the wheels, therefore the minimum weight required on the driving-wheels is $7 \times 4{\cdot}37 = 30{\cdot}6$ tons.

(1) The greatest retarding force with continuous brakes is $120 \times \frac{1}{7}$ tons. Hence, if t = number of seconds necessary to bring the train to rest, the impulse $120 \times \frac{1}{7} \times t = \dfrac{120}{32{\cdot}2} \times \frac{88}{1} \times \frac{2}{3}$, the momentum in ton and second units. Hence—

$$t = \frac{7 \times 88 \times 2}{3 \times 32{\cdot}2} = 12{\cdot}75 \text{ seconds}$$

(2) If the brakes are on the driving-wheels only, the retarding force will be restricted to $\frac{1}{7}$ of 30·6 tons, *i.e.* to 4·37 tons, which was the accelerating force, and consequently the time required to come to rest will be the same as that required to accelerate, *i.e.* 50 seconds.

Example 3.—A square-threaded screw 2 inches mean diameter has two threads per inch of length, the coefficient of friction between the screw and nut being 0·02. Find the horizontal force applied at the circumference of the screw necessary to lift a weight of 3 tons.

The pitch of the screw is $\frac{1}{2}$ inch.

$$\text{If } \alpha = \text{angle of the screw, } \tan \alpha = \frac{0 \cdot 5}{2\pi} = 0 \cdot 0794$$

$$\text{and if } \lambda = \text{angle of friction, } \tan \lambda = 0 \cdot 02$$

Let P = force necessary in tons.

$$\frac{P}{3} = \tan(\alpha + \lambda) = \frac{\tan \alpha + \tan \lambda}{1 - \tan \alpha \tan \lambda} = \frac{0 \cdot 0794 + 0 \cdot 02}{1 - 0 \cdot 0794 \times 0 \cdot 02}$$

$$= \frac{0 \cdot 0994}{0 \cdot 9984} = 0 \cdot 09956$$

hence P = 0·2987 ton

Examples XI.

1. A block of iron weighing 11 lbs. can be pulled along a horizontal wooden plank by a horizontal force of 1·7 lbs. What is the coefficient of friction between the iron and the plank? What is the greatest angle to the horizontal through which the plank can be tilted without the block of iron sliding off?

2. What is the least force required to drag a block of stone weighing 20 lbs. along a horizontal path, and what is its direction, the coefficient of friction between the stone and the path being 0·15?

3. What horizontal force is required to start a body weighing 15 lbs. moving up a plane inclined 30° to the horizontal, the coefficient of friction between the body and the plane being 0·25?

4. Find the least force in magnitude and direction required to drag a log up a road inclined 15° to the horizontal if the coefficient of friction between the log and the road is 0·4.

5. With a coefficient friction 0·2, what must be the inclination of a plane to the horizontal if the work done by the minimum force in dragging 10 lbs. a vertical distance of 3 feet up the plane is 60 foot lbs.?

6. A shaft bearing 6 inches diameter carries a dead load of 3 tons, and the shaft makes 80 rotations per minute. The coefficient of friction between the shaft and bearing is 0·012. Find the horse-power absorbed in friction in the bearing.

7. If a brake shoe is pressed against the outside of a wheel with a force of 5 tons, and the coefficient of friction between the wheel and the brake is 0·3, find the horse-power absorbed by the brake if the wheel is travelling at a uniform speed of 20 miles per hour.

8. A stationary rope passes over part of the circumference of a rotating pulley, and acts as a brake upon it. The tension of the tight end of the rope is 120 lbs., and that of the slack end 25 lbs., the difference being due to the frictional force exerted tangentially to the pulley rim. If the pulley makes 170 rotations per minute, and is 2 feet 6 inches diameter, find the horse-power absorbed.

9. A block of iron weighing 14 lbs. is drawn along a horizontal wooden table by a weight of 4 lbs. hanging vertically, and connected to the block of iron by a string passing over a light pulley. If the coefficient of friction between the iron and the table is 0·15, find the acceleration of the block and the tension of the string.

10. A locomotive has a total weight of 30 tons on the driving wheels, and the coefficient of friction between the wheels and rails is 0·15. What is the greatest pull it can exert on a train? Assuming the engine to be sufficiently powerful to exert this pull, how long will it take the train to attain a speed of 20 miles per hour if the gross weight is 120 tons, and the resistances amount to 20 lbs. per ton?

11. A square-threaded screw, 1·25 inches mean diameter, has five threads per inch of length. Find the force in the direction of the axis exerted by the screw when turned against a resistance, by a handle which exerts a force equivalent to 500 lbs. at the circumference of the screw, the coefficient of friction being 0·08.

CHAPTER VI

STATICS OF RIGID BODIES

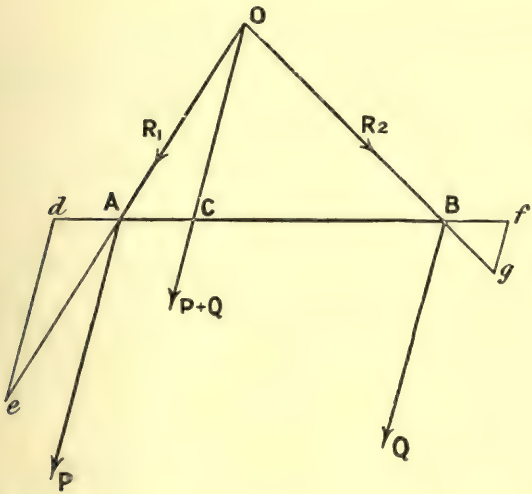
85. THE previous chapter dealt with bodies of very small dimensions, or with others under such conditions that all the forces acting upon them were concurrent.

In general, however, the forces keeping a rigid body in equilibrium will not have lines of action all passing through one point. Before stating the conditions of equilibrium of a rigid body, it will be necessary to consider various systems of non-concurrent forces. We shall assume that two intersecting forces may be replaced by their geometric sum acting through the point of intersection of their lines of action; also that a force may be considered to act at any point in its line of action. Its point of application makes no difference to the equilibrium of the body, although upon it will generally depend the distribution of internal forces in the body. With the internal forces or stresses in the body we are not at present concerned.

86. Composition of Parallel Forces.—The following constructions are somewhat artificial, but we shall immediately from them find a simpler method of calculating the same results.

To find the resultant and equilibrant of any two given like parallel forces, *i.e.* two acting in the same direction. Let P and Q (Fig. 72) be the forces of given magnitudes. Draw any line, AB, to meet the lines of action of P and Q in A and B respectively. At A and B introduce two equal and opposite forces, S, acting in the line AB, and applied one at A and the other at B. Compound S and P at A by adding the vectors A*d* and *de*, which give a vector A*e*, representing R_1, the resultant

of S and P. Similarly, compound S and Q at B by adding the vectors B*f* and *fg*, which give a vector sum B*g*, representing R_2, the resultant of Q and S. Produce the lines of action of R_1 and R_2 to meet in O, and transfer both forces to O. Now resolve R_1 and R_2 at O into their components again, and we



Vector *de* represents P.
Vector *fg* represents Q.
Vectors A*d* and B*f* represent equal and opposite forces S.

FIG. 72.

have left two equal and opposite forces, S, which have a resultant nil, and a force P + Q acting in the same direction as P and Q along OC, a line parallel to the lines of action of P and Q. If a force P + Q acts in the line CO in the opposite direction to P and Q, it balances their resultant, and therefore it will balance P and Q, *i.e.* it is their equilibrant.

Let the line of action of the resultant P + Q cut AB in C.

Since AOC and A*ed* are similar triangles—

$$\frac{CA}{OC} = \frac{Ad}{ae} = \frac{S}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and since BOC and B*gf* are similar triangles—

$$\frac{CB}{OC} = \frac{Bf}{fg} = \frac{S}{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

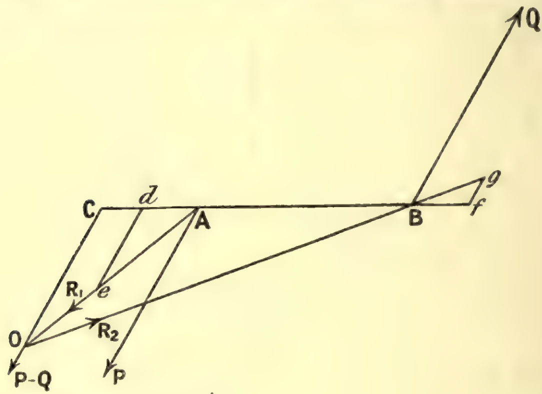
and dividing equation (2) by equation (1)—

$$\frac{CB}{CA} = \frac{P}{Q}$$

or the point C divides the line AB in the inverse ratio of the magnitude of the two forces; and similarly the line of action OC of the resultant P + Q divides any line meeting the lines of action of P and Q in the inverse ratio of the forces.

To find the resultant of any two given unlike parallel forces, *i.e.* two acting in opposite directions.

Let one of the forces, P, be greater than the other, Q (Fig. 73). By introducing equal and opposite forces, S, at A



Vector *de* represents P.
Vector *fg* represents Q.
Vectors A*d* and B*f* represent equal and opposite forces S.

FIG. 73.

and B, and proceeding exactly as before, we get a force P − Q acting at O, its line of action cutting AB produced in C. Since AOC and A*ed* are similar triangles—

$$\frac{CA}{OC} = \frac{Ad}{de} = \frac{S}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and since BOC and B*gf* are similar triangles—

$$\frac{CB}{CO} = \frac{Bf}{fg} = \frac{S}{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Dividing equation (4) by equation (3)—

$$\frac{CB}{CA} = \frac{P}{Q}$$

or the line of action of the resultant P − Q divides the line AB (and any other line cutting the lines of action of P and Q) externally, in the inverse ratio of the two forces, cutting it beyond the line of the greater force. If a force of magnitude P − Q acts in the line CO in the opposite direction to that of P (*i.e.* in the same direction as Q), it balances the resultant of P and Q, and therefore it will balance P and Q; *i.e.* it is their equilibrant.

This process fails if the two unlike forces are equal. The resultants R_1 and R_2 are then also parallel, and the point of intersection O is non-existent. The two equal unlike parallel forces are not equivalent to, or replaceable by, any single force, but form what is called a "couple."

More than two parallel forces might be compounded by successive applications of this method, first to one pair, then to the resultant and a third force, and so on. We shall, however, investigate later a simpler method of compounding several parallel forces.

87. Resolution into Parallel Components.—In the last article we replaced two parallel forces, P and Q, acting at points A and B, by a single force parallel to P and Q, acting at a point C in AB, the position of C being such that it divides AB inversely as the magnitudes of the forces P and Q. Similarly, a single force may be replaced by two parallel forces

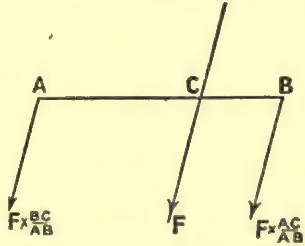


FIG. 74.—Resolution into two like parallel components.

acting through any two given points. Let F (Fig. 74) be the single force, and A and B be the two given points. Join AB

and let C be the point in which AB cuts the line of action of F. If, as in Fig. 74, A and B are on opposite sides of F, then F may be replaced by parallel forces in the same direction as F, at A and B, the magnitudes of which have a sum F, and which are in the inverse ratio of their distances from C, viz. a force $F \times \frac{CB}{AB}$ at A, and a force $F \times \frac{AC}{AB}$ at B. The parallel equilibrants or balancing forces of F acting at A and B are then forces $F \times \frac{CB}{AB}$ and $F \times \frac{AC}{AB}$ respectively, acting in the opposite direction to that of the force F.

If A and B are on the same side of the line of action of the force F (Fig. 75), then F may be replaced by forces at A and B,

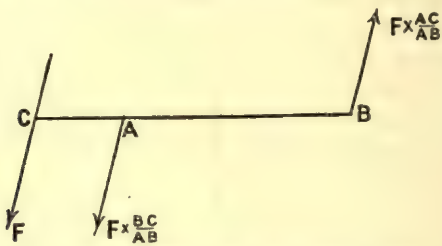


FIG. 75.—Resolution into two unlike parallel components.

the magnitudes of which have a difference F, the larger force acting through the nearer point A, and in the same direction as the force F, the smaller force acting through the further point B, and in the opposite direction to the force F, and the magnitudes being in the inverse ratio of the distances of the forces from C, viz. a force $F \times \frac{CB}{AB}$ at A, in the direction of F, and an opposite force $F \times \frac{AC}{AB}$ at B.

The *equilibrants* of F at A and B will be $F \times \frac{CB}{AB}$ in the opposite direction to that of F, and $F \times \frac{AC}{AB}$ in the direction of F, respectively.

As an example of the parallel equilibrants through two points, A and B, on either side of the line of action of a force, we may take the vertical upward reactions at the supports of a beam due to a load concentrated at some place on the beam.

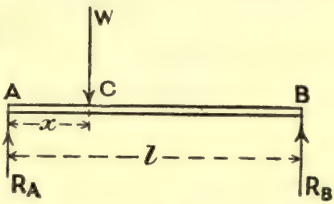


FIG. 76.

Let W lbs. (Fig. 76) be the load at a point C on a beam of span l feet, C being x feet from A, the left-hand support, and therefore $l - x$ feet from the right-hand support, B.

Let R_A be the supporting force or reaction at A;

R_B be the supporting force or reaction at B.

$$\text{Then } R_A = W \times \frac{BC}{AB} = W\frac{l - x}{l} \text{ lbs.}$$

$$\text{and } R_B = W \times \frac{AC}{AB} = W\frac{x}{l} \text{ lbs.}$$

More complicated examples of the same kind where there is more than one load will generally be solved by a slightly different method.

88. Moments.—The moment of a force F lbs. about a fixed point, O, was measured (Art. 56) by the product $F \times d$ lb.-feet, where d was the perpendicular distance in feet from O to the line of action of F. Let ON (Fig. 77) be the perpendicular from O on to the line of action of a force F.

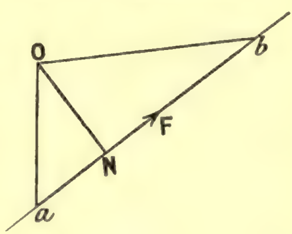


FIG. 77.

Set off a vector ab on the line of action of F to represent F. Then the product $ab \cdot ON$, which is twice the area of the triangle Oab, is proportional to the moment of F about O. Some convention as to signs of clockwise and contra-clockwise moments (Art. 56) must be adopted. If the moment of F about O is contra-clockwise, *i.e.* if O lies to the left of the line

of action of F viewed in the direction of the force, it is usual to reckon the moment and the area Oab representing it as positive, and if clockwise to reckon them as negative.

89. Moment of a Resultant Force.—This, about any point in the plane of the resultant and its components, is equal to the algebraic sum of the moments of the components. Let O (Fig. 78) be any point in the plane of two forces, P and Q, the lines of action of which intersect at A. Draw Od parallel to the force P, cutting the line of action of Q in c. Let the vector Ac represent the force Q, and set off Ab in the line of action of P to represent P on the same scale, *i.e.* such that Ab = A$c \times \frac{P}{Q}$.

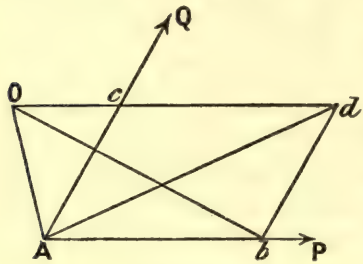


FIG. 78.

Complete the parallelogram Abdc. Then the vector Ad = Ac + cd = Ac + Ab, and represents the resultant R, of P and Q.

Now, the moment of P about O is represented by twice the area of triangle AOb (Art. 88), and the moment of Q about O is represented by twice the area of triangle AOc, and the moment of R about O is represented by twice the area of triangle AOd.

But the area AOd = area Acd + area AOc
= area Abd + area AOc

Abd and Acd being each half of the parallelogram Abdc; hence area AOd = area AOb + AOc, since AOb and Abd are between the same parallels ; or—

twice area AOd = twice area AOb + twice area AOc.

and these three quantities represent respectively the moments of R, P, and Q about O. Hence the moment of R about O is equal to the sum of the moments of P and Q about that point.

If O is to the right of one of the forces instead of to the left

of both, as it is in Fig. 78, there will be a slight modification in sign; *e.g.* if O is to the right of the line of action of Q and to the left of R and P, the area AO*c* and the moment of Q about O will be negative, but the theorem will remain true for the algebraic sum of the moments.

Next let the forces P and Q be parallel (Fig. 79). Draw

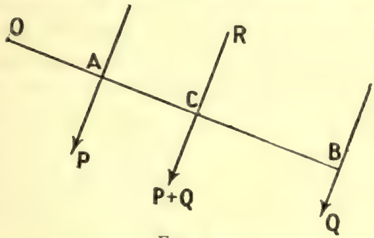


FIG. 79.

a line AB from O perpendicular to the lines of action of P and Q, cutting them in A and B respectively. Then the resultant R, which is equal to P + Q, cuts AB in C such that $\frac{BC}{AC} = \frac{P}{Q}$.

$$\text{Then } P \, . \, AC = Q \, . \, BC$$

The sum of moments of P and Q about O is $P \, . \, OA + Q \, . \, OB$, and this is equal to $P(OC - AC) + Q(OC + CB)$, which is equal to $(P + Q)OC - P \, . \, AC + Q \, . \, CB = (P + Q)OC$, since $P \, . \, AC = Q \, . \, CB$.

And $(P + Q)OC$ is the moment of the resultant R about O. Hence the moment of the resultant is equal to the sum of moments of the two component forces. The figure will need modification if the point O lies between the lines of action of P and Q, and their moments about O will be of opposite sign, but the moment of R will remain equal to the algebraic sum of those of P and Q. The same remark applies to the figure for two unlike parallel forces.

The force equal and opposite to the resultant, *i.e.* the equilibrant, of the two forces (whether parallel or intersecting) has a moment of equal magnitude and opposite sign to that of the resultant (Art. 88), and *therefore the equilibrant has a moment*

about any point in the plane of the forces, of equal magnitude and of opposite sign to the moments of the forces which it balances. In other words, the algebraic sum of the moments of any two forces and their equilibrant about any point in their plane is zero.

90. Moment of Forces in Equilibrium.—If several forces, all in the same plane, act upon a body, the resultant of any two has about any point O in the plane a moment equal to that of the two forces (Art. 89). Applying the same theorem to a third force and the resultant of the first two, the moment of their resultant (*i.e.* the resultant of the first three original forces) is equal to that of the three forces, and so on. By successive applications of the same theorem, it is obvious that the moment of the final resultant of all the forces about any point in their plane is equal to the sum of the moments of all the separate forces about that point, whether the forces be all parallel or inclined one to another.

If the body is in equilibrium, the resultant force upon it in any plane is zero, and therefore *the algebraic sum of the moments of all the separate forces about any point in the plane is zero.* This fact gives a method of finding one or two unknown forces acting on a body in equilibrium, particularly when their lines of action are known. When more than one force is unknown, the clockwise and contra-clockwise moments about any point in the line of action of one of the unknown forces may most conveniently be dealt with, for the moment of a force about any point in its line of action is zero.

The Principle of Moments, *i.e.* the principle of equation of the algebraic sum of moments of all forces in a plane acting on a body in equilibrium to zero, or equation of the clockwise to the contra-clockwise moments, will be most clearly understood from the three examples at the end of this article.

Levers.—A lever is a bar free to turn about one fixed point and capable of exerting some force due to the exertion of an effort on some other part of the bar. The bar may be of any shape, and the fixed point, which is called the fulcrum, may be in any position. When an effort applied to the lever is just sufficient to overcome some given opposing force, the lever has just passed a condition of equilibrium, and the relation

between the effort, the force exerted by the lever, and the reaction at the fulcrum may be found by the principle of moments.

Example 1.—A roof-frame is supported by two vertical walls 20 feet apart at points A and B on the same level. The line of the resultant load of 4 tons on the frame cuts the line AB 8 feet from A, at an angle of 75° to the horizontal, as shown in Fig. 80. The supporting force at the point B is a vertical one. Find its amount.

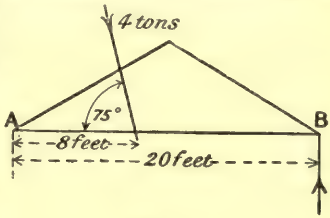


FIG. 80.

The supporting force through the point A is unknown, but its moment about A is zero. Hence the clockwise moment of the 4-ton resultant must balance the contra-clockwise moment of the vertical supporting force R_B at B.

Equating the magnitudes of the moments—

$$4 \times 8 \sin 75° = 20 \times R_B \text{ (tons-feet)}$$

$$\text{therefore } R_B = \frac{32 \sin 75°}{20} = 1{\cdot}6 \times 0{\cdot}9659 = 1{\cdot}545 \text{ tons}$$

Example 2.—A light horizontal beam of 12-feet span carries loads of 7 cwt., 6 cwt., and 9 cwt. at distances of 1 foot, 5 feet, and 10 feet respectively from the left-hand end. Find the reactions of the supports of the beam.

If we take moments about the left-hand end A (Fig. 81), the

AC = 1 foot.
AD = 5 feet.
AE = 10 feet.
AB = 12 feet.

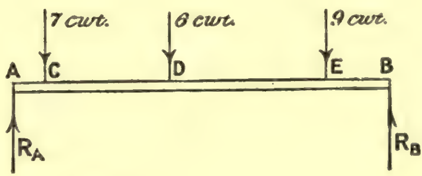


FIG. 81.

vertical loads have a clockwise tendency, and the moment of the reaction R_B at B is contra-clockwise ; hence—

$$R_B \times 12 = (7 \times 1) + (6 \times 5) + (9 \times 10)$$
$$12R_B = 7 + 30 + 90 = 127$$
$$R_B = \tfrac{127}{12} = 10{\cdot}58\dot{3} \text{ cwt.}$$

R_A, the supporting force at A, may be found by an equation of moments about B. Or since—

$$R_B + R_A = 7 + 6 + 9 = 22 \text{ cwt.}$$
$$R_A = 22 - 10{\cdot}58\dot{3} = 11{\cdot}41\dot{6} \text{ cwt.}$$

Example 3.—An L-shaped lever, of which the long arm is 18 inches long and the short one 10 inches, has its fulcrum at the right angle. The effort exerted on the end of the long arm is 20 lbs., inclined 30° to the arm. The short arm is kept from moving by a cord attached to its end and perpendicular to its length. Find the tension of the chord.

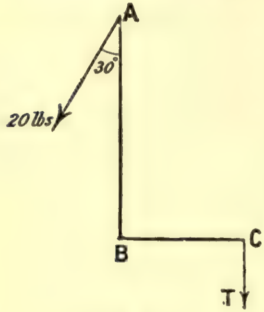


Fig. 82.

Let T be the tension of the string in pounds.

Then, taking moments about B (Fig. 82), since the unknown reaction of the hinge or fulcrum has no moment about that point—

$$AB \sin 30^\circ \times 20 = BC \times T$$
$$18 \times \tfrac{1}{2} \times 20 = 10 \times T$$
$$T = 18 \text{ lbs.}$$

Examples XII.

1. A post 12 feet high stands vertically on the ground. Attached to the top is a rope, inclined downwards and making an angle of 25° with the horizontal. Find what horizontal force, applied to the post 5 feet above the ground, will be necessary to keep it upright when the rope is pulled with a force of 120 lbs.

2. Four forces of 5, 7, 3, and 4 lbs. act along the respective directions AB, BC, DC, and AD of a square, ABCD. Two other forces act, one in CA, and the other through D. Find their amounts if the six forces keep a body in equilibrium.

3. A beam of 15-feet span carries loads of 3 tons, ½ ton, 5 tons, and 1 ton, at distances of 4, 6, 9 and 13 feet respectively from the left-hand end. Find the pressure on the supports at each end of the beam, which weighs ¾ ton.

4. A beam 20 feet long rests on two supports 16 feet apart, and overhangs the left-hand support 3 feet, and the right-hand support by 1 foot. It carries a load of 5 tons at the left-hand end of the beam, and one of 7 tons midway between the supports. The weight of the beam, which may be looked upon as a load at its centre, is 1 ton. Find the reactions at the

supports, *i.e.* the supporting forces. What upward vertical force at the right-hand end of the beam would be necessary to tilt the beam?

5. A straight crowbar, AB, 40 inches long, rests on a fulcrum, C, near to A, and a force of 80 lbs. applied at B lifts a weight of 3000 lbs. at A. Find the distance AC.

6. A beam 10 feet long rests upon supports at its ends, and carries a load of 7 cwt. 3 feet from one end. Where must a second load of 19 cwt. be placed in order that the pressures on the two supports may be equal?

91. Couples.—In Art. 86 it was stated that two equal unlike parallel forces are not replaceable by a single resultant force; they cannot then be balanced by a single force. Such a system is called a *couple*, and the perpendicular distance between the lines of action of the two forces is called the *arm* of the couple. Thus, in Fig. 83, if two equal and opposite forces F lbs. act at A and B perpendicular to the line AB, they form a couple, and the length AB is called the arm of the couple.

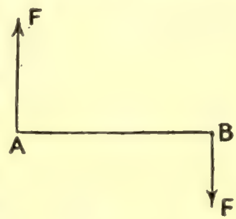


FIG. 83.

92. Moment of a Couple.—This is the tendency to produce rotation, and is measured by the product of one of the forces forming the couple and the arm of the couple; *e.g.* if the two equal and opposite forces forming the couple are each forces of 5 lbs., and the distance apart of their lines of action is 3 feet, the moment of the couple is 5×3, or 15 *lb.-feet;* or in Fig. 83, the moment of the couple is $F \times AB$ in suitable units.

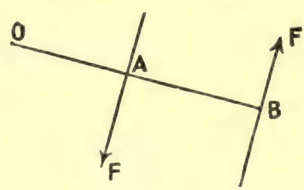


FIG. 84.

The sum of the moments of the forces of a couple is the same about any point O in their plane. Let O (Fig. 84) be any point. Draw a line OAB perpendicular to the lines of action of the forces and meeting them in A and B. Then the total (contra-clockwise) moment of the two forces about O is—

$$F \cdot OB - F \cdot OA = F(OB - OA) = F \cdot AB$$

This is the value, already stated, of the moment of the couple, and is independent of the position of O.

A couple is either of clockwise or contra-clockwise tendency, and its moment about any point in its plane is of the same tendency (viewed from the same aspect) and of the same magnitude.

93. Equivalent Couples.—Any two couples in a plane having the same moment are equivalent if they are of the same sign or turning tendency, *i.e.* either both clockwise or both contra-clockwise; or, if the couples are equal in magnitude and of opposite sign, they balance or neutralise one another. The latter form of the statement is very simply proved. Let the forces F, F (Fig. 85) constitute a contra-clockwise couple, and the forces F′, F′ constitute a clock-wise couple having a moment of the same magnitude. Let the lines of action of F, F and those of F′, F′ intersect in A, B, C, and D, and let AE be the perpendicular from A on BC, and CG the perpendicular from C on AB. Then, the moments of the two couples being equal—

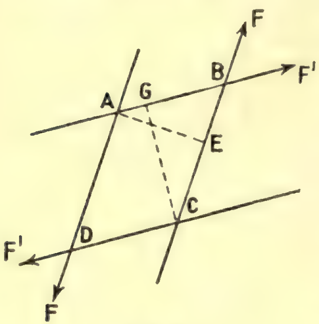


FIG. 85.

$$F \times AE = F' \times GC$$
$$F \times AB \sin A\hat{B}C = F' \times CB \sin A\hat{B}C$$
$$F \times AB = F' \times CB$$
$$\frac{F}{F'} = \frac{CB}{AB}$$

Hence CB and AB may, as vectors, fully represent F and F′ respectively, acting at B. And since ABCD is a parallelogram, CD = AB, and the resultant or vector sum of F and F′ is in the line DB, acting through B in the direction DB.

Similarly, the forces F and F′ acting at D have an equal and opposite resultant acting through D in the direction BD. These two equal and opposite forces in the line of B and D balance, hence the two couples balance.

It has been assumed here that the lines of action of F and F′ intersect; if they do not, equal and opposite forces in the same straight line may, for the purpose of demonstration, be introduced and compounded with the forces of one couple without affecting the moment of that couple or the equilibrium of any system of which it forms a part.

94. Addition of Couples. — The resultant of several couples in the same plane and of given moments is a couple the moment of which is equal to the sum of the moments of the several couples.

Any couple may be replaced by its equivalent couple having an arm of length AB (Fig. 93) and forces F_1, F_1, provided $F_1 \times AB =$ moment of the couple.

Similarly, a second couple may be replaced by a couple of arm AB and forces F_2, F_2, provided $F_2 \times AB$ is equal to the moment of this second couple. In this way clockwise couples must be replaced by clockwise couples of arm AB, and contra-clockwise couples by contra-clockwise couples of arm AB, until finally we have a couple of moment—

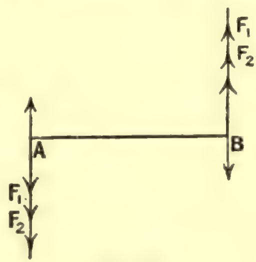


Fig. 86.

$$
\begin{aligned}
(F_1 + F_2 + F_3 + \ldots \text{ etc.})AB &= F_1 \times AB + F_2 \times AB + F_3 \times AB + \ldots \text{ etc.} \\
&= \text{algebraic sum of moments of the given couples}
\end{aligned}
$$

the proper sign being given to the various forces.

95. Reduction of a System of Co-planar Forces.— A system of forces all in the same plane is equivalent to (1) a single resultant force, or (2) a couple, or (3) a system in equilibrium, which may be looked upon as a special case of (1), viz. a single resultant of magnitude zero.

Any two forces of the system which intersect may be replaced by a single force equal to their geometric sum acting through the point of intersection. Continuing the same process

of compounding successive forces with the resultants of others as far as possible, the system reduces to either a single resultant, including the case of a zero resultant, or to a number of parallel forces. In the latter case the parallel forces may be compounded by applying the rules of Art. 86, and reduced to either a single resultant (including a zero resultant) or to a couple. Finally, then, the system must reduce to (1) a single resultant, or (2) a couple, or (3) the system is in equilibrium.

96. Conditions of Equilibrium of a System of Forces in One Plane.—If such a system of forces is in equilibrium, the geometric or vector sum of all the forces must be zero, or, in other words, the force polygon must be a closed one, for otherwise the resultant would be (Art. 95) a single force represented by the vector sum of the separate forces.

Also, if the system is in equilibrium (*i.e.* has a zero resultant), the algebraic sum of all the moments of the forces about any point in their plane is zero (Art. 90). These are all the conditions which are necessary, as is evident from Art. 95, but they may be conveniently stated as three conditions, which are sufficient—

(1) and (2) The sum of the components in each of two directions must be zero (a single resultant has a zero component in *one* direction, viz. that perpendicular to its line of action).

(3) The sum of the moments of all the forces about *one* point in the plane is zero.

If conditions (1) and (2) are fulfilled the system cannot have a single resultant (Art. 75), and if condition (3) is fulfilled it cannot reduce to a couple (Art. 92), and therefore it must reduce to a zero resultant (Art. 95), *i.e.* the system must be in equilibrium.

These three conditions are obviously necessary, and they have just been shown to be sufficient, but it should be remembered that the algebraic sum of the moments of all the forces about *every* point in the plane is zero. The above three conditions provide for three equations between the magnitudes of the forces of a system in equilibrium and their relative positions, and from these equations three unknown quantities may be found if all other details of the system be known.

97. Solution of Statical Problems.—In finding the forces acting upon a system of rigid bodies in equilibrium, it should be remembered that each body is in itself in equilibrium, and therefore we can obtain three relations (Art. 96) between the forces acting upon it, viz. we can write three equations by stating in algebraic form the three conditions of equilibrium; that is, we may resolve all the forces in two directions, preferably at right angles, and equate the components in opposite directions, or equate the algebraic sums to zero, and we may equate the clockwise and contra-clockwise moments about any point, or equate the algebraic sum of moments to zero.

The moment about *every* point in the plane of a system of co-planar forces in equilibrium is zero, and sometimes it is more convenient to consider the moments about two points and only resolve the forces in one direction, or to take moments about three points and not resolve the forces. If more than three equations are formed by taking moments about other points, they will be found to be not independent and really a repetition of the relations expressed in the three equations formed. Some directions of resolution are more convenient than others, *e.g.* by resolving perpendicular to some unknown force, no component of that force enters into the equation so formed. Again, an unknown force may be eliminated in an equation of moments by taking the moments about some point in its line of action, about which it will have a zero moment.

"Smooth" Bodies.—An absolutely smooth body would be one the reaction of which, on any body pressing against it, would have no frictional component, *i.e.* would be normal to the surface of contact, the angle of friction (Art. 79) being zero. No actual body would fulfil such a condition, but it often happens that a body is so smooth that any frictional force it may exert upon a second body is so small in comparison with other forces acting upon that body as to be quite negligible, *e.g.* if a ladder with one end on a rough floor rest against a horizontal round steel shaft, such as is used to transmit power in workshops, the reaction of the shaft on the ladder might

without serious error be considered perpendicular to the length of the ladder, *i.e.* normal to the cylindrical surface of the shaft.

Example 1.—A horizontal rod 3 feet long has a hole in one end, A, through which a horizontal pin passes forming a hinge. The other end, B, rests on a smooth roller at the same level. Forces of 7, 9, and 5 lbs. act upon the rod, their lines of action, which are in the same vertical plane, intersecting it at distances of 11, 16, and 27 inches respectively from A, and making acute angles of 30°, 75°, and 45° respectively with AB, the first two sloping downwards towards A, and the third sloping downwards towards B, as shown in Fig. 87. Find the magnitude of the supporting forces on the rod at A and B.

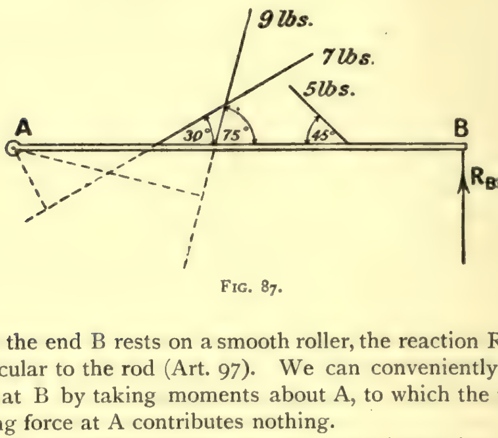


Fig. 87.

Since the end B rests on a smooth roller, the reaction R_B at B is perpendicular to the rod (Art. 97). We can conveniently find this reaction at B by taking moments about A, to which the unknown supporting force at A contributes nothing.

The total clockwise moment about A in lb.-inches is—

$$\left.\begin{array}{l} 7 \times 11 \sin 30° + 9 \times 16 \sin \\ 75° + 5 \times 27 \sin 45° \end{array}\right\} = 77 \times 0·5 + 144 \times 0·966 + 135 \times 0·707$$
$$= 270·2 \text{ lb.-inches}$$

The total contra-clockwise moment about A is $R_B \times 36$. Equating the moments of opposite sign—

$$R_B \times 36 = 270·2 \text{ lb.-inches}$$
$$R_B = \frac{270·2}{36} = 7·5 \text{ lbs.}$$

The remaining force R_A through A may be found by drawing to scale an open vector polygon with sides representing the forces 7, 9, 5, and 7·5 lbs. (R_B); the closing side then represents R_A.

Or we may find R_A by resolving all the forces, say, horizontally and vertically. Let H_A be the horizontal component of R_A estimated positively to the right, and V_A its vertical component upwards. Then, by Art. 96, the total horizontal component of all the forces is zero; hence—

$$H_A - 7 \cos 30^\circ - 9 \cos 75^\circ + 5 \cos 45^\circ = 0$$
$$H_A = 7 \times 0{\cdot}866 + 9 \times 0{\cdot}259 - 5 \times 0{\cdot}707 = 4{\cdot}85 \text{ lbs.}$$

Also the total vertical component is zero, hence—

$$V_A - 7 \sin 30^\circ - 9 \sin 75^\circ - 5 \sin 45^\circ + 7{\cdot}5 = 0$$
$$V_A = 7 \times \tfrac{1}{2} + 9 \times 0{\cdot}966 + 5 \times 0{\cdot}707 - 7{\cdot}5 = 8{\cdot}23 \text{ lbs.}$$

Compounding these two rectangular components of R_A—

$$R_A = \sqrt{\{(4{\cdot}85)^2 + (8{\cdot}23)^2\}} \text{ (Art. 75)}$$
$$R_A = \sqrt{91{\cdot}2} = 9{\cdot}54 \text{ lbs.}$$

Example 2.—ABCD is a square, each side being 17·8 inches, and E is the middle point of AB. Forces of 7, 8, 12, 5, 9, and 6 lbs. act on a body in the lines and directions AB, EC, BC, BD,

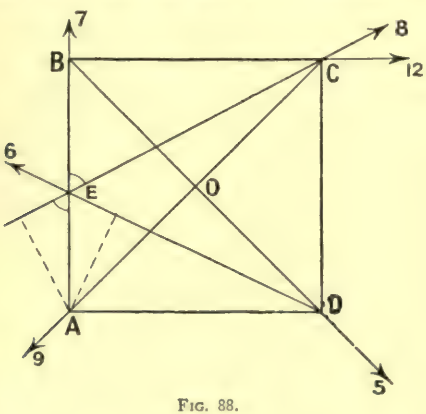


FIG. 88.

CA, and DE respectively. Find the magnitude, and position with respect to ABCD, of the single force required to keep the body in equilibrium.

Let F be the required force ;

H_A be the component of F in the direction AD ;

V_A be the component of F in the direction AB ;

p be the perpendicular distance in inches of the force from A.

Then, resolving in direction AD, the algebraic total component being zero—

$$\left.\begin{aligned} H_A + 8\cos O\hat{E}C + 12 + 5\cos 45^\circ - 9\cos 45^\circ \\ - 6\cos E\hat{D}A \end{aligned}\right\} = 0$$

$$H_A + 8 \times \frac{2}{\sqrt{5}} + 12 - 4 \times \frac{1}{\sqrt{2}} - 6 \times \frac{2}{\sqrt{5}} = 0$$

$$H_A + (2 \times 0{\cdot}895) + 12 - 4 \times 0{\cdot}707 = 0$$

$$H_A = -10{\cdot}96 \text{ lbs.}$$

Resolving in direction AB—

$$\left.\begin{aligned} V_A + 7 + 8\cos B\hat{E}C - 5\cos 45^\circ - 9\cos 45^\circ \\ + 6\cos A\hat{E}D \end{aligned}\right\} = 0$$

$$V_A + 7 + 14 \times \frac{1}{\sqrt{5}} - 14 \times \frac{1}{\sqrt{2}} = 0$$

$$V_A = -7 - 6{\cdot}26 + 9{\cdot}90 = -3{\cdot}36$$

$$\text{then } F = \sqrt{\{(10{\cdot}96)^2 + (3{\cdot}36)^2\}} = 11{\cdot}46 \text{ lbs.}$$

and is inclined to AD at an angle the tangent of which is—

$$\frac{-3{\cdot}36}{-10{\cdot}96} = 0{\cdot}3065$$

i.e. at an angle $180 + 17^\circ$ or 197°.

Its position remains to be found. We may take moments about any point, say A. Let p be reckoned positive if F has a contra-clockwise moment about A.

$$\left.\begin{aligned} 11{\cdot}46 \times p + 6 \times AD \sin A\hat{D}E - 5 \times OA - 12 \\ \times AB - 8 \times AE \sin B\hat{E}C \end{aligned}\right\} = 0$$

$$11{\cdot}46p = -\frac{106{\cdot}8}{\sqrt{5}} + \frac{89}{\sqrt{2}} + 213{\cdot}6 + \frac{142{\cdot}4}{\sqrt{5}} = 0$$

$$p = \frac{292{\cdot}3}{11{\cdot}46} = 25{\cdot}51 \text{ inches}$$

This completes the specification of the force F, which makes an angle 197° with AD and passes $25{\cdot}51$ inches from A, so as to have a contra-clockwise moment about A. The position of F is shown in Fig. 89.

The force might be specified as making 197° with AD and cutting it at a distance $25.51 \div \sin 197°$ or -86.5 inches from A; *i.e.* 86·5 inches to the left of A.

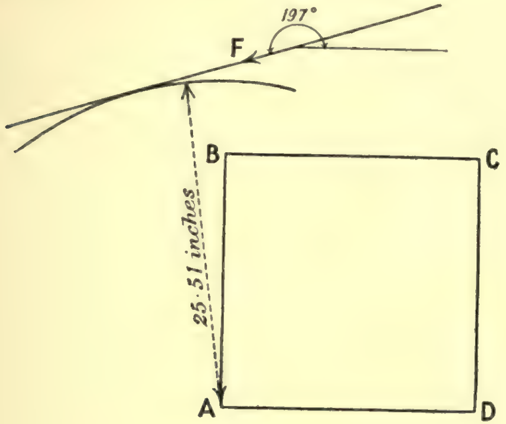


FIG. 89.

98. Method of Sections.—The principles of the preceding article may be applied to find the forces acting in the members of a structure consisting of separate pieces jointed together. If the structure be divided by an imaginary plane of section into two parts, either part may be looked upon as a body in equilibrium under certain forces, some of which are the forces exerted by members cut by the plane of section.

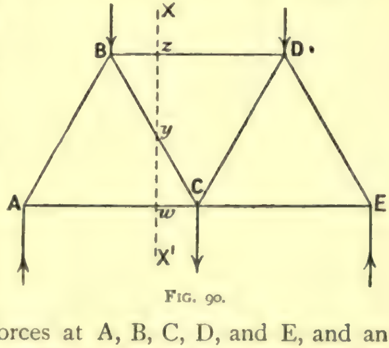


FIG. 90.

For example, if a hinged frame such as ABCDE (Fig. 90) is in equilibrium under given forces at A, B, C, D, and E, and an imaginary plane of section XX′ perpendicular to the plane of the structure be taken, then the portion AB*zyw* is in equilibrium

under the forces at A and B, and the forces exerted upon it by the remaining part of the structure, viz. the forces in the bars BD, BC, and AC. This method of sections is often the simplest way of finding the forces in the members of a jointed structure.

Example.—One end of a girder made up of bars jointed together is shown in Fig. 91. Vertical loads of 3 tons and 5 tons are carried at B and C respectively, and the vertical supporting force at H is 12 tons. The sloping bars are inclined at 60° to the horizontal. Find the forces in the bars CD, CE, and FE.

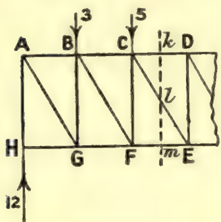


Fig. 91.

The portion of the girder ACFH cut off by the vertical plane *klm* is in equilibrium under the action of the loads at B and C, the supporting force at H, and the forces exerted by the bars CD, CE, and FE on the joints at C and F. Resolving these forces vertically, the forces in CD and FE have no vertical component, hence the *downward* vertical component force exerted by CE on the left-hand end of the girder is equal to the excess upward force of the remaining three, *i.e.* $12 - 3 - 5 = 4$ tons; hence—

$$\text{Force in CE} \times \cos 30° = 4 \text{ tons}$$

$$\text{or force in CE} = 4 \times \frac{2}{\sqrt{3}} = 4{\cdot}61 \text{ tons}$$

This, being positive, acts downwards on the left-hand end, *i.e.* it acts towards E, or the bar CE *pulls* at the joint C, hence the bar CE is in tension to the amount of 4·61 tons. To find the force in bar FE, take a vertical section plane through C or indefinitely near to C, and just on the right hand of it. Then, taking moments about C and reckoning clockwise moments positive—

$$12 \times \text{AC} - 3 \times \text{BC} + \sqrt{3} \times \text{FE} \times (\text{force in FE}) = 0$$

$$12 \times 2 - 3 \times 1 + \sqrt{3} \times (\text{force in FE}) = 0$$

$$\text{and force in FE} = -\frac{21}{\sqrt{3}} = -12{\cdot}12 \text{ tons}$$

The negative sign indicates that the force in FE acts on F in the opposite direction to that in which it would have a clockwise moment about C, *i.e.* the force *pulls* at the joint F; hence the member is in *tension* to the extent of 12·12 tons.

Similarly, taking say clockwise moments about E, the force in CD is found to be a push of 14·42 tons towards C, *i.e.* CD has a compressive force of 14·42 tons in it, as follows:—

$$12 \times 3 - 3 \times 2 - 5 \times 1 + \sqrt{3}\,(\text{force in CD}) = 0$$
$$\text{force in CD} = -14{\cdot}42$$

99. Rigid Body kept in Equilibrium by Three Forces.—If three forces keep a body in equilibrium, they either all pass through one point (*i.e.* are concurrent) or are all parallel. For unless all three forces are parallel two must intersect, and these are replaceable by a single resultant acting through their point of intersection. This resultant cannot balance the third force unless they are equal and opposite and in the same straight line, in which case the third force passes through the intersection of the other two, and the three forces are concurrent.

The fact of either parallelism or concurrence of the three forces simplifies problems on equilibrium under three forces by fixing the position of an unknown force, since its line of action intersects those of the other two forces at their intersection. The magnitude of the forces can be found by a triangle of forces, or by the method of resolution into rectangular components.

Statical problems can generally be solved in various ways, some being best solved by one method, and others by different methods. In the following example four methods of solution are indicated, three of which depend directly upon the fact that the three forces are concurrent, which gives a simple method of determining the direction of the reaction of the rough ground.

Example 1.—A ladder 18 feet long rests with its upper end against a smooth vertical wall, and its lower end on rough ground 7 feet from the foot of the wall. The weight of the ladder is 40 lbs., which may be looked upon as a vertical force halfway along the length of the ladder. Find the magnitude and direction of the forces exerted by the wall and the ground on the ladder.

The weight of 40 lbs. acts vertically through C (Fig. 92), and the reaction of the wall F_1 is perpendicular to the wall (Art. 97). These two forces intersect at D. The only remaining force, F_2, on

the ladder is the pressure which the ground exerts on it at B. This must act through D also (Art. 99), and therefore its line of action must be BD. F_1 may be found by an equation of the moments about B.

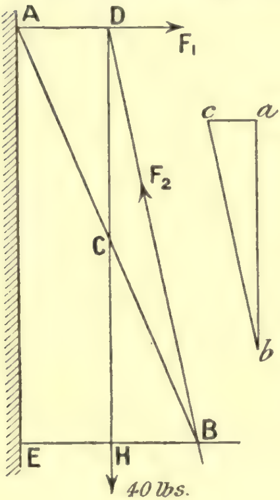


FIG. 92.

$$F_1 \times AE = 40 \times \tfrac{1}{2}BE.$$
$$F_1 \times \sqrt{(18^2 - 7^2)} = 40 \times \tfrac{7}{2}$$
$$F_1 = \frac{140}{\sqrt{(275)}}$$
$$= 8{\cdot}44 \text{ lbs.}$$

And since F_2 balances the horizontal force of 8·44 lbs. and a vertical force of 40 lbs.—

$$F_2 = \sqrt{\{(8{\cdot}44)^2 + 40^2\}} = 40{\cdot}8 \text{ lbs.}$$

and is inclined to EB at an angle $E\hat{B}D$, the tangent of which is—

$$\frac{AE}{\frac{1}{2}EB} = \frac{2 \times \sqrt{(275)}}{7} = 4{\cdot}74$$

which is the tangent of 78·1°.

A second method of solving the problem consists in drawing a vector triangle, *abc* (Fig. 92), representing by its vector sides F_1, F_2, and 40 lbs. The 40-lb. force *ab* being set off to scale, and *bc* and *ca* being drawn parallel to F_2 and F_1 respectively, and the magnitudes then measured to the same scale. A third method consists (without drawing to scale) of solving the triangle *abc* trigonometrically, thus—

$$F_1 : F_2 : 40 = ca : cb : ab$$
$$= HB : BD : HD$$
$$= 3{\cdot}5 : \sqrt{\{(3{\cdot}5)^2 + 275\}} : \sqrt{(275)}$$

from which F_1 and F_2 may be easily calculated, viz.—

$$F_1 = \frac{40 \times 7}{2 \times \sqrt{275}} = 8{\cdot}44 \text{ lbs.}$$

$$F_2 = 40 \times \frac{\sqrt{287}}{\sqrt{275}} = 40{\cdot}8 \text{ lbs.}$$

Fourthly, the problem might be solved very simply by resolving the forces F_1 and F_2 and 40 lbs. horizontally and vertically, as in

this particular case the 40-lb. weight has no component in the direction of F_1, and must exactly equal in magnitude the vertical component of F_2; the horizontal component of F_2 must also be just equal to the magnitude of F_1.

Example 2.—A light bar, AB, 20 inches long, is hinged at A so as to be free to move in a vertical plane. The end B is supported by a cord, BC, so placed that the angle $A\hat{B}C$ is $145°$ and AB is horizontal. A weight of 7 lbs. is hung on the bar at a point D in AB 13 inches from A. Find the tension in the cord and the pressure of the rod on the hinge.

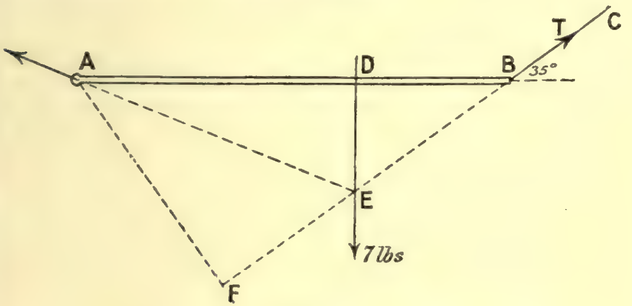


FIG. 93.

Let T be the tension in the cord, and P be the pressure on the hinge.

Taking moments about A, through which P passes (Fig. 93)—

$$T \times AF = 7 \times AD$$
$$T \times 20 \sin 35° = 7 \times 13$$
$$11{\cdot}47T = 91$$
$$T = 7{\cdot}94 \text{ lbs.}$$

The remaining force on the bar is the reaction of the hinge, which is equal and opposite to the pressure P of the bar on the hinge.

The vertical upward component of this is $7 - T \sin 35°$ $= 2{\cdot}45$ lbs., and the horizontal component is $T \cos 35° = 6{\cdot}5$ lbs.

$$\text{Hence } P = \sqrt{(6{\cdot}5)^2 + (2{\cdot}45)^2} = 6{\cdot}93 \text{ lbs.}$$

The tangent of the angle $D\hat{A}E$ is $\frac{2{\cdot}45}{6{\cdot}5} = 0{\cdot}377$, corresponding to an angle of $20° \, 40'$.

The pressure of the bar on the hinge is then 6·93 lbs. in a

direction, AE, inclined downwards to the bar and making an angle 20° 40′ with its length.

EXAMPLES XIII.

1. A trap door 3 feet square is held at an inclination of 30° to (and above) the horizontal plane through its hinges by a cord attached to the middle of the side opposite the hinges. The other end of the cord, which is 5 feet long, is attached to a hook vertically above the middle point of the hinged side of the door. Find the tension in the cord, and the direction and magnitude of the pressure between the door and its hinges, the weight of the door being 50 lbs., which may be taken as acting at the centre of the door.

2. A ladder 20 feet long rests on rough ground, leaning against a rough vertical wall, and makes an angle of 60° to the horizontal. The weight of the ladder is 60 lbs., and this may be taken as acting at a point 9 feet from the lower end. The coefficient of friction between the ladder and ground is 0·25. If the ladder is just about to slip downwards, find the coefficient of friction between it and the wall.

3. A ladder, the weight of which may be taken as acting at its centre, rests against a vertical wall with its lower end on the ground. The coefficient of friction between the ladder and the ground is $\frac{1}{3}$, and that between the ladder and the wall $\frac{1}{4}$. What is the greatest angle to the vertical at which the ladder will rest?

4. A rod 3 feet long is hinged by a horizontal pin at one end, and supported on a horizontal roller at the other. A force of 20 lbs. inclined 45° to the rod acts upon it at a point 21 inches from the hinged end. Find the amount of the reactions on the rod at the hinge and at the free end.

5. A triangular roof-frame ABC has a horizontal span AC of 40 feet, and the angle at the apex B is 120°, AB and BC being of equal length. The roof is hinged at A, and simply supported on rollers at C. The loads it bears are as follow: (1) A force of 4000 lbs. midway along and perpendicular to AB; (2) a vertical load of 1500 lbs. at B; and (3) a vertical load of 1400 lbs. midway between B and C. Find the reactions or supporting forces on the roof at A and C.

6. Draw a 2-inch square ABCD, and find the middle point E of AB. Forces of 17, 10, 8, 7, and 20 lbs. act in the directions CB, AB, EC, ED, and BD respectively. Find the magnitude, direction, and position of the force required to balance these. Where does it cut the line AD, and what angle does it make with the direction AD?

7. A triangular roof-frame ABC has a span AC of 30 feet. AB is 15 feet, and BC is 24 feet. A force of 2 tons acts normally to AB at its middle point, and another force of 1 ton, perpendicular to AB, acts at B. There is also a vertical load of 5 tons acting downward at B. If the supporting force at A is a vertical one, find its magnitude and the magnitude and direction of the supporting force at C.

8. A jointed roof-frame, ABCDE, is shown in Fig. 94. AB and BC are inclined to the horizontal at 30°, EB and DB are inclined at 45° to the

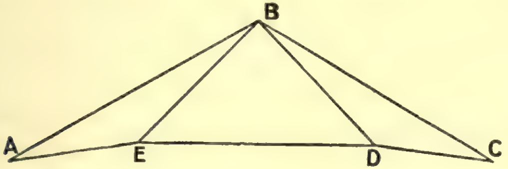


FIG. 94.

horizontal. The span AC is 40 feet, and B is 10 feet vertically above ED. Vertical downward loads of 2 tons each are carried at B, at E, and at D. Find by the method of sections the forces in the members AB, EB, and ED.

9. A jointed structure, ACD . . . LMB (Fig. 95) is built up of bars all

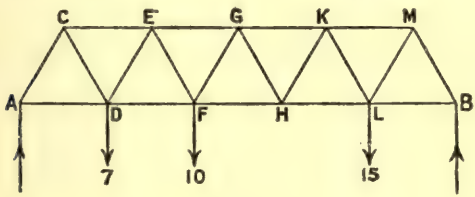


FIG. 95.

of equal length, and carries loads of 7, 10, and 15 tons at D, F and L respectively. Find by the method of sections the forces on the bars EF, EG, and DF.

CHAPTER VII

CENTRE OF INERTIA OR MASS—CENTRE OF GRAVITY

100. Centre of a System of Parallel Forces.—Let A, B, C, D, E, etc. (Fig. 96), be points at which parallel forces F_1, F_2, F_3, F_4, F_5, etc., respectively act. The position of the resultant force may be found by applying successively the rule

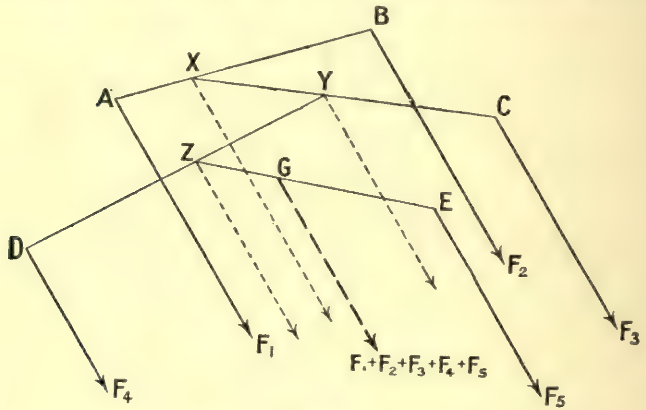


FIG. 96.

of Art. 86. Thus F_1 and F_2 may be replaced by a force $F_1 + F_2$, at a point X in AB such that $\frac{AX}{XB} = \frac{F_2}{F_1}$ (Art. 86).

This force acting at X, and the force F_3 acting at C, may be replaced by a force $F_1 + F_2 + F_3$ at a point Y in CX such that $\frac{XY}{YC} = \frac{F_3}{F_1 + F_2}$ (Art. 86).

Proceeding in this way to combine the resultant of several forces with one more force, the whole system may be replaced by a force equal to the algebraic sum of the several forces acting at some point G. It may be noticed that the positions of the points X, Y, Z, and G depend only upon the positions of the points of application A, B, C, D, and E of the several forces and the magnitude of the forces, and are independent of the directions of the forces provided they are parallel. The point of application G of the resultant is called the *centre* of the parallel forces F_1, F_2, F_3, F_4, and F_5 acting through A, B, C, D, and E respectively, whatever direction those parallel forces may have.

101. Centre of Mass.—If every particle of matter in a body be acted upon by a force proportional to its mass, and all the forces be parallel, the centre of such a system of forces (Art. 100) is called the centre of mass or centre of inertia of the body. It is quite independent of the direction of the parallel forces, as we have seen in Art. 100.

Centre of Gravity.—The attraction which the earth exerts upon every particle of a body is directed towards the centre of the earth, and in bodies of sizes which are small compared to that of the earth, these forces may be looked upon as parallel forces. Hence these gravitational forces have a centre, and this is called the *centre of gravity* of the body; it is, of course, the same point as the centre of mass.

The resultant of the gravitational forces on all the particles of a body is called its weight, and in the case of rigid bodies it acts through the point G, the centre of gravity, whatever the position of the body. A change of position of the body is equivalent to a change in direction of the parallel gravitational forces on its parts, and we have seen (Art. 100) that the centre of such a system of forces is independent of their direction. We now proceed to find the centres of gravity in a number of special cases.

102. Centre of gravity of two particles of given weights at a given distance apart, or of two bodies the centres of gravity and weights of which are given.

Let A and B (Fig. 97) be the positions of the two particles

(or centres of gravity of two bodies) of weights w_1 and w_2

FIG. 97.

respectively. The centre of gravity G is (Art. 86) in AB at such a point that—

$$\frac{\text{GA}}{\text{GB}} = \frac{w_2}{w_1}$$

$$\text{or GA} = \frac{w_2}{w_1 + w_2} \cdot \text{AB}$$

$$\text{and GB} = \frac{w_1}{w_1 + w_2} \cdot \text{AB}$$

In the case of two equal weights, AG = GB = $\frac{1}{2}$AB.

A convenient method of finding the point G graphically may be noticed. Set off from A (Fig. 98) a line AC, making any angle with AB (preferably at right angles), and proportional to w_2 to any scale; from B set off a line BD parallel to AC on the opposite side of AB, and proportional to w_1 to the same scale that AC represents w_2. Join CD. Then the intersection of CD with AB determines the point G. The proof follows simply from the similarity of the triangles ACG and BDG.

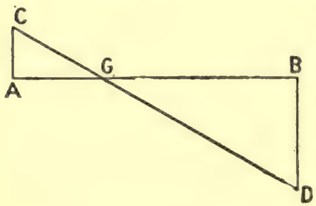


FIG. 98.

103. Uniform Straight Thin Rod.—Let AB (Fig. 99) be the uniform straight rod of length AB: it may be supposed to be divided into pairs of particles of equal weight situated at equal distances from the middle point G of the rod, since there will be as many such particles between A and G as between G and B. The c.g. (centre of gravity) of each pair, such as the particles at a and b, is midway between them (Art. 102), viz. at the middle point of the rod, G, hence the c.g. of the whole rod is at its middle point, G.

FIG. 99.

104. Uniform Triangular Plate or Lamina.—The term *centre of gravity of an area* is often used to denote the c.g. of a thin lamina of uniform material cut in the shape of the particular area concerned.

We may suppose the lamina ABC (Fig. 100) divided into an indefinitely large number of strips parallel to the base AC. The c.g. of each strip, such as PQ, is at its middle point (Art. 103), and every c.g. is therefore in the median BB,′ *i.e.* the line joining B to the mid-point B′ of the base AC. Hence the c.g. of the whole triangular lamina is in the median BB′. Similarly, the c.g. of the lamina is in the medians AA′ and CC′. Hence the c.g. of the triangle is at G, the intersection of the three medians, which are concurrent, meeting at a point distant from any vertex of the triangle by $\frac{2}{3}$ of the median through it. The perpendicular distance of G from any side of the triangle is $\frac{1}{3}$ of the perpendicular distance of the opposite vertex from that side.

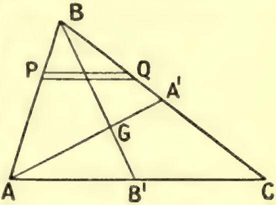


FIG. 100.

Note that the c.g. of the triangular area ABC coincides with that of three equal particles placed at A, B, and C. For those at A and C are statically equivalent to two at B′, and the c.g. of two at B′ and one at B is at G, which divides BB′ in the ratio 2 : 1, or such at B′G = $\frac{1}{3}$BB′ (Art. 102).

Uniform Parallelogram.—If a lamina be cut in the shape of a parallelogram, ABCD (Fig. 101), the c.g. of the triangle ABC is in OB, and that of the triangle ADC is in OD, therefore the c.g. of the whole is in BD. Similarly it is in AC, and therefore it is at the intersection O.

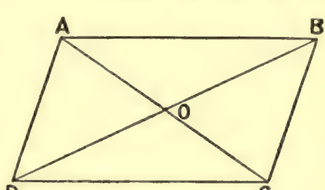


FIG. 101.

105. Rectilinear Figures in General.—The c.g. of any lamina with straight sides may be found by dividing its area up into triangles, and finding the c.g. and area of each triangle.

Thus, in Fig. 102, if G_1, G_2, and G_3 are the centres of gravity of the triangles ABE, EBD, and DBC respectively, the c.g. of the area ABDE is at G_4, which divides the length G_1G_2 inversely as the weights of the triangles AEB and EDB, and therefore inversely as their areas. Similarly, the c.g. G of the whole figure ABCDE divides G_3G_4 inversely as the areas of the figures ABDE and BCD. The inverse division of the lines G_1G_2 and of G_3G_4 may in practice be performed by the graphical method of Art. 102.

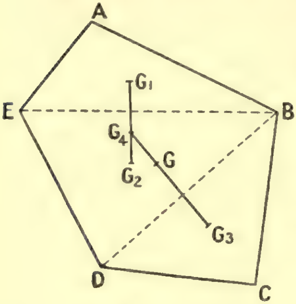


FIG. 102.

106. Symmetrical Figures.—If a plane figure has an axis of symmetry, *i.e.* if a straight line can be drawn dividing it

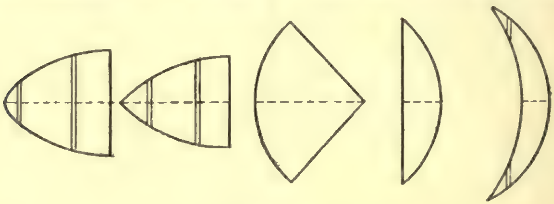


FIG. 103.

into two exactly similar halves, the c.g. of the area of the figure lies in the axis of symmetry. For the area can be divided into indefinitely narrow strips, the c.g. of each of which is in the axis of symmetry (see Fig. 103). If a figure has two or more axes of

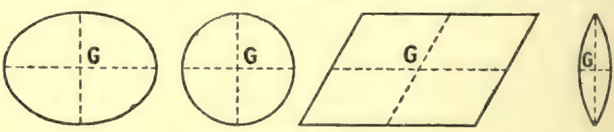


FIG. 104.

symmetry, the c.g. must lie in each, hence it is at their intersection, *e.g.* the c.g. of a circular area is at its centre. Other examples, which sufficiently explain themselves, are shown in Fig. 104.

107. Lamina or Solid from which a Part has been removed.—Fig. 105 represents a lamina from which a piece, B, has been cut. The centre of gravity of the whole lamina, including the piece B, is at G, and the c.g. of the removed portion B is at g. The area of the remaining piece A is a units, and that of the piece B is b units. It is required to find the c.g. of the remaining piece A.

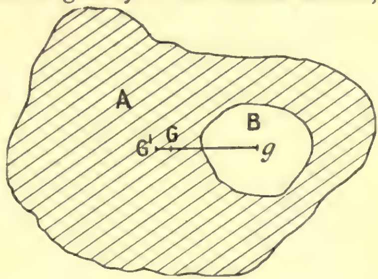


FIG. 105.

Let G′ be the required c.g.; then G is the c.g. of two bodies the centres of gravity of which are at G′ and g, and which are proportional to a and b respectively. Hence G is in the line G′g, and is such that—

$$GG' : Gg :: b : a \text{ (Art. 102)}$$

$$\text{or } GG' = \frac{b}{a} . Gg$$

That is, the c.g. G′ of the piece A is in the same straight line gG as the two centres of gravity of the whole and the part B, at $\frac{b}{a}$ times their distance apart beyond the c.g. of the whole lamina. The point G′ divides the line Gg externally in the ratio $\frac{b}{a+b}$, or $G'G : G'g :: b : a + b$.

The same method is applicable if A is part of a solid from which a part B has been removed, provided a represents the weight of the part A, and b that of the part B.

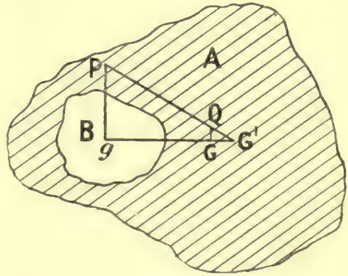


FIG. 106.

Graphical Construction.—The c.g. of the part A may be found as follows: from g draw a line gP (Fig. 106) at any angle (preferably at right angles) to Gg and proportional to $a + b$. From G

draw GQ parallel to gP and proportional to b. Join PQ, and produce to meet gG produced in G′. Then G′ is the c.g. of the part A.

108. Symmetrical Solids of Uniform Material.—If a solid is symmetrical about one plane, *i.e.* if it can be divided by a plane into two exactly similar halves, the c.g. evidently lies in the plane, for the solid can be divided into laminæ the

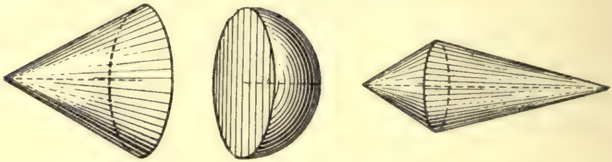


FIG. 107.

c.g. of each of which is in the plane of symmetry. Similarly, if the solid has two planes of symmetry, the c.g. must lie in the intersection of the two planes, which is an axis of the solid, as in Fig. 107.

If a solid has three planes of symmetry, the line of intersection of any two of them meets the third in the c.g., which is

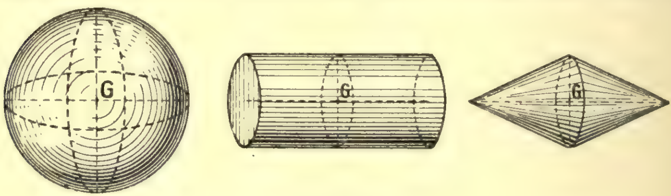


FIG. 108.

a point common to all three planes, *e.g.* the sphere, cylinder, etc. (see Fig. 108).

109. Four Equal Particles not in the Same Plane.—Let ABCD (Fig. 109) be the positions of the four equal particles. Join ABCD, forming a triangular pyramid or tetrahedron. The c.g. of the three particles at A, B, and C is at D′, the c.g. of the triangle ABC (Art. 104). Hence the c.g. of the four particles is at G in DD′, and is such that—

$$D'G : GD = 1 : 3 \text{ (Art. 102)}$$
$$\text{or } D'G = \tfrac{1}{4} DD'$$

Similarly, the c.g. of the four particles is in AA′, BB′, and CC′, the lines (which are concurrent) joining A, B, and C to the centres of gravity of the triangles BCD, ACD, and ABD respectively. The distance of the c.g. from any face of the tetrahedron is $\frac{1}{4}$ of the perpendicular distance of the opposite vertex from that face.

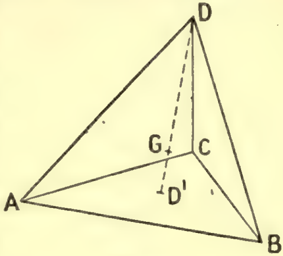


FIG. 109.

110. Triangular Pyramid or Tetrahedron of Uniform Material.—Let ABCD (Fig. 110) be the triangular pyramid. Suppose the solid divided into indefinitely thin plates, such as *abc*, by planes parallel to the face ABC. Let D′ be the c.g. of the area ABC. Then DD′ will intersect the plate *abc* at its c.g., viz. at *d*, and the c.g. of every plate, and therefore of the whole solid, will be in DD′. Similarly, it will be in AA′, BB′, and CC′, where A′, B′, and C′ are the centres of gravity of the triangles BCD, CDA, and DAB respectively. Hence the centre of gravity of the whole solid coincides with that of four equal particles placed at its vertices (Art. 109), and it is in DD′, and distant $\frac{1}{4}$ DD′ from D′, in CC′ and $\frac{1}{4}$ CC′ from C′, and so on. It is, therefore, also distant from any face, $\frac{1}{4}$ of the perpendicular distance of the opposite vertex from that face.

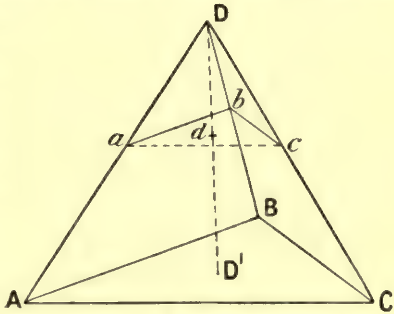


FIG. 110.

111. Uniform Pyramid or Cone on a Plane Base.—If V (Fig. 111) is the vertex of the cone, and V′ the c.g. of the base of the cone, the c.g. of any parallel section or lamina into which the solid may be divided by plates parallel to the base, will be in VV′. Also if the base be divided into an indefinitely large number of indefinitely small triangles, the solid is made up of

an indefinitely large number of triangular pyramids having the triangles as bases and a common vertex, V. The c.g. of each small pyramid is distant from V $\frac{3}{4}$ of the distance from its base to V. Hence the centres of gravity of all the pyramids lie in a plane parallel to the base, and distant from the vertex, $\frac{3}{4}$ of the altitude of the cone.

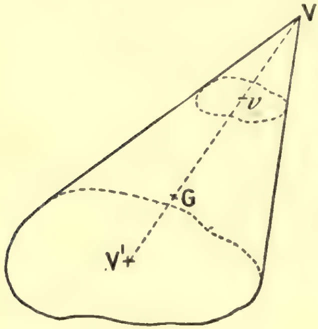


FIG. 111.

The c.g. of a right circular cone is therefore in its axis, which is the intersection of two planes of symmetry (Art. 108), and its distance from the base is $\frac{1}{4}$ the height of the cone, or its distance from the vertex is $\frac{3}{4}$ of the height of the cone.

Example 1.—A solid consists of a right circular cylinder 3 feet long, and a right cone of altitude 2 feet, the base coinciding with one end of the cylinder. The cylinder and cone are made of the same uniform material. Find the c.g. of the solid.

If r = radius of the cylinder in feet—

$$\frac{\text{the volume of cylinder}}{\text{volume of cone}} = \frac{\pi r^2 \times 3}{\pi r^2 \times \frac{1}{3} \times 2} = \frac{9}{2}$$

hence the weight of the cylinder is 4·5 times that of the cone.

The c.g. of the cylinder is at A (Fig. 112), the mid-point of its axis (Art. 108), *i.e.* 1·5 feet from the plane of the base of the cone.

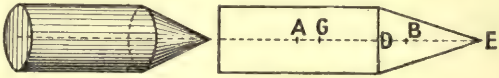


FIG. 112.

The c.g. of the cone is at B, $\frac{1}{4}$ of the altitude from the base (Art. 111), *i.e.* 0·5 foot from the common base of the cylinder and cone. Hence—

$$AB = AD + DB = 1{\cdot}5 + 0{\cdot}5 = 2 \text{ feet}$$

And G is therefore in AB, at a distance $\frac{2}{2+9}$. AB from A (Art. 102), *i.e.* AG = $\frac{2}{11}$ of 2 feet = $\frac{4}{11}$ foot, or $4{\cdot}\dot{3}\dot{6}$ inches.

Example 2.—A quadrilateral consists of two isosceles triangles on opposite sides of a base 8 inches long. The larger triangle has two equal sides each 7 inches long, and the smaller has its vertex 3 inches from the 8-inch base. Find the distance of the c.g. of the quadrilateral from its 8-inch diagonal.

Let ABCD (Fig. 113) be the quadrilateral, AC being the 8-inch diagonal, of which E is the mid-point ; then—

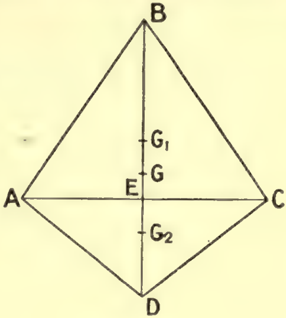


FIG. 113.

$$ED = 3 \text{ inches}$$
$$EB = \sqrt{7^2 - 4^2} = \sqrt{33} = 5{\cdot}745 \text{ inches}$$

The c.g. of the triangle ABC is in EB and $\frac{1}{3}$ EB from E ; or, if G_1 is the c.g.—

$$EG_1 = \frac{5{\cdot}745}{3} = 1{\cdot}915 \text{ inches}$$

Similarly, if G_2 is the c.g. of the triangle ADC—

$$EG_2 = \tfrac{1}{3} \text{ of } 3 \text{ inches} = 1 \text{ inch}$$
$$\text{therefore } G_1G_2 = 1{\cdot}915 + 1 = 2{\cdot}915 \text{ inches}$$

This length is divided by G, the c.g. of the quadrilateral, so that—

$$\frac{G_2G}{G_1G} = \frac{\text{area of triangle ABC}}{\text{area of triangle ACD}} = \frac{BE}{ED} = \frac{1{\cdot}915}{1}$$
$$\frac{G_2G}{G_1G_2} = \frac{1{\cdot}915}{1 + 1{\cdot}915} = \frac{1{\cdot}915}{2{\cdot}915}$$
$$G_2G = 1{\cdot}915 \text{ inches}$$
$$\text{and } EG = G_2G - G_2E = 1{\cdot}915 - 1 = 0{\cdot}915 \text{ inch}$$

which is the distance of the c.g. from the 8-inch diagonal.

Example 3.—A pulley weighs 25 lbs., and it is found that the c.g. is 0·024 inch from the centre of the pulley. The pulley is required to have its c.g. at the geometrical centre of the rim, and to correct the error in its position a hole is drilled in the pulley with its centre 6 inches from the pulley centre and in the same diameter as the wrongly placed c.g. How much metal should be removed by drilling ?

Let x be the weight of metal to be removed, in pounds.

Then, in Fig. 114, OA being 6 inches and OG 0·024 inch, the removed weight x lbs. having its c.g. at A, and the remaining

Fig. 114.

$25 - x$ lbs. having its c.g. at O, the c.g. G of the two together divides OA, so that—

$$\frac{\mathrm{OG}}{\mathrm{GA}} = \frac{x}{25 - x}$$

$$\text{or } \frac{\mathrm{OG}}{\mathrm{OA}} = \frac{x}{25}$$

$$\text{hence } x = \frac{25 \times \mathrm{OG}}{\mathrm{OA}} = 25 \times \frac{0·024}{6} = 0·1 \text{ lb.}$$

Examples XIV.

1. A uniform beam weighing 180 lbs. is 12 feet long. It carries a load of 1000 lbs. uniformly spread over 7 feet of its length, beginning 1 foot from one end and extending to a point 4 feet from the other. Find at what part of the beam a single prop would be sufficient to support it.

2. A lever 4 feet long, weighing 15 lbs., but of varying cross-section, is kept in equilibrium on a knife-edge midway between its ends by the application of a downward force of 1·3 lbs. at its lighter end. How far is the c.g. of the lever from the knife-edge?

3. The heavy lever of a testing machine weighs 2500 lbs., and is poised horizontally on a knife-edge. It sustains a downward pull of 4 tons 3 inches from the knife-edge, and carries a load of 1 ton on the same side of the knife-edge and 36 inches from it. How far is the c.g. of the lever from the knife-edge?

4. A table in the shape of an equilateral triangle, ABC, of 5 feet sides, has various articles placed upon its top, and the legs at A, B, and C then exert pressures of 30, 36, and 40 lbs. respectively on the floor. Determine the position of the c.g. of the table loaded, and state its horizontal distances from the sides AB and BC.

5. Weights of 7, 9, and 12 lbs. are placed in the vertices A, B, and C respectively of a triangular plate of metal weighing 10 lbs., the dimensions of which are, AB 16 inches, AC 16 inches, and BC 11 inches. Find the c.g. of the plate and weights, and state its distances from AB and BC.

6. One-eighth of a board 2 feet square is removed by a straight saw-cut through the middle points of two adjacent sides. Determine the distance of the c.g. of the remaining portion from the saw-cut. If the whole board before part was removed weighed 16 lbs., what vertical upward force

applied at the corner diagonally opposite the saw cut would be sufficient to tilt the remaining $\frac{7}{8}$ of the board out of a horizontal position, if it turned about the line of the saw-cut as a hinge?

7. An isosceles triangle, ABC, having AB 10 inches, AC 10 inches, and base BC 4 inches long, has a triangular portion cut off by a line DE, parallel to the base BC, and 7·5 inches from it, meeting AB and AC in D and E respectively. Find the c.g. of the trapezium BDEC, and state its distance from the base BC.

8. The lever of a testing-machine is 15 feet long, and is poised on a knife-edge 5 feet from one end and 10 feet from the other, and in a horizontal line, above and below which the beam is symmetrical. The beam is 16 inches deep at the knife-edge, and tapers uniformly to depths of 9 inches at each end; the width of the beam is the same throughout its length. Find the distance of the c.g. of the beam from the knife-edge.

9. A retaining wall 5 feet high is vertical in front and 9 inches thick at the top. The back of the wall slopes uniformly, so that the thickness of the wall at the base is 2 feet 3 inches. Find the c.g. of the cross-section of the wall, and state its horizontal distance from the vertical face of the wall.

10. What is the moment of the weight of the wall in Question 9 per foot length, about the back edge of the base, the weight of the material being 120 lbs. per cubic foot? What uniform horizontal pressure per square foot acting on the vertical face of the wall would be sufficient to turn it over bodily about the back edge of the base?

11. The casting for a gas-engine piston may be taken approximately as a hollow cylinder of uniform thickness of shell and one flat end of uniform thickness. Find the c.g. of such a casting if the external diameter is 8 inches, the thickness of shell $\frac{3}{4}$ inch, that of the end 3 inches, and the length over all 20 inches. State its distance from the open end.

12. A solid circular cone stands on a base 14 inches diameter, and its altitude is 20 inches. From the top of this a cone is cut having a base 3·5 inches diameter, by a plane parallel to the base. Find the distance of the c.g. of the remaining frustum of the cone from its base.

13. Suppose that in the rough, the metal for making a gun consists of a frustum of a cone, 10 feet long, 8 inches diameter at one end, and 6 inches at the other, through which there is a cylindrical hole 3 inches diameter, the axes of the barrel and cone being coincident. How far from the larger end must this piece of metal be slung on a crane in order to remain horizontal when lifted?

14. A pulley weighing 40 lbs. has its c.g. 0·04 inch from its centre. This defect is to be rectified by drilling a hole on the heavy side of the pulley, with its centre 9 inches from the centre of the pulley and in the radial direction of the centre of gravity. What weight of metal should be drilled out?

15. A cast-iron pulley weighs 45 lbs., and has its c.g. 0·035 inch from its centre. In order to make the c.g. coincide with the centre of the

pulley, metal is added to the light side at a distance of 8 inches from the centre of the pulley and in line with the c.g. What additional weight is required in this position? If the weight is added by drilling a hole in the pulley and then filling it up to the original surface with lead, how much iron should be removed, the specific gravity of lead being 11·35, and that of iron being 7·5?

112. Distance from a Fixed Line of the Centre of Gravity of Two Particles, or Two Bodies, the Centres of Gravity of which are given.

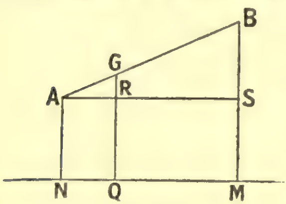


FIG. 115.

Let A (Fig. 115) be the position of a particle of weight w_1, and let B be that of a particle of weight w_2, or, if the two bodies are of finite size, let A and B be the positions of their centres of gravity. Then the centre of gravity of the two weights w_1 and w_2 is at G in AB such that—

$$\frac{AG}{GB} = \frac{w_2}{w_1} \text{ (Art. 102)}$$

$$\text{or } AG = \frac{w_2}{w_1 + w_2} . AB$$

$$\text{and } GB = \frac{w_1}{w_1 + w_2} . AB$$

Let the distances of A, B, and G from the line NM be x_1, x_2, and $\bar{x}$ respectively, the line NM being in a plane through the line AB. Then AN = x_1, BM = x_2, and GQ = $\bar{x}$.

$$\text{Now, } \frac{GR}{BS} = \frac{AG}{AB} = \frac{w_2}{w_1 + w_2}$$

$$\text{or } GR = \frac{w_2}{w_1 + w_2} . BS$$

$$\text{and GQ or } \bar{x} = RQ + GR = AN + \frac{w_2}{w_1 + w_2} BS$$

$$\text{hence } \bar{x} = x_1 + \frac{w_2}{w_1 + w_2}(x_2 - x_1) = \frac{x_1 w_1 + x_2 w_2}{w_1 + w_2}$$

Distance of the c.g. from a Plane.—If x_1 and x_2 are the respective distances of A and B from any plane, then NM

may be looked upon as the line joining the feet of perpendiculars from A and B upon that plane. Then the distance $\bar{x}$ of G from that plane is—

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This length $\bar{x}$ is also called the mean distance of the two bodies or particles from the plane.

113. Distance of the c.g. of Several Bodies or of One Complex Body from a Plane.

Let A, B, C, D, and E (Fig. 116) be the positions of 5 particles weighing w_1, w_2, w_3, w_4, and w_5 respectively, or the

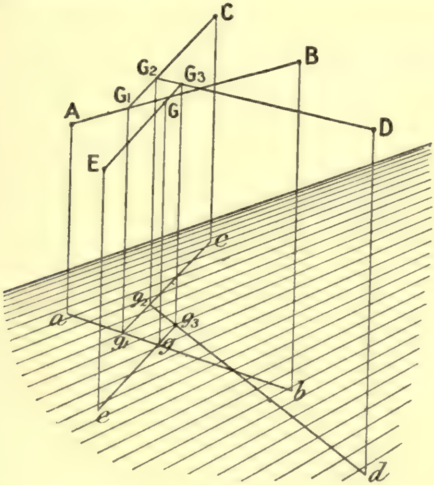


FIG. 116.

centres of gravity of five bodies (or parts of one body) of those weights.

Let the distances of A, B, C, D, and E from some fixed plane be x_1, x_2, x_3, x_4, and x_5 respectively, and let the weights in those positions be w_1, w_2, w_3, w_4, and w_5 respectively. It is required to find the distance $\bar{x}$ of the c.g. of these five weights from the plane. We may conveniently consider the plane to

be a horizontal one, but this is not essential; then x_1, x_2, x_3, x_4, and x_5 are the vertical heights of A, B, C, D, and E respectively above the plane. Let a, b, c, d, and e be the projections or feet of perpendiculars from A, B, C, D, and E respectively on the plane, so that Aa, Bb, Cc, Dd, and Ee are equal to x_1, x_2, x_3, x_4, and x_5 respectively.

Let G_1 be the c.g. of w_1 and w_2, and let g_1 be its projection by a vertical line on the plane; then—

$$G_1g_1 = \frac{w_1x_1 + w_2x_2}{w_1 + w_2} \text{ (Art. 112. (1))}$$

Let G_2 be the c.g. of $(w_1 + w_2)$ and w_3, and let g_2 be its projection by a vertical line on the plane; then G_2 divides G_1C so that—

$$G_1G_2 = \frac{w_3}{(w_1 + w_2) + w_3}G_1C$$

$$\text{and } G_2g_2 = \frac{(w_1 + w_2)G_1g_1 + w_3x_3}{w_1 + w_2 + w_3} \text{ (Art. 112. (1))}$$

and substituting the above value of G_1g_1—

$$G_2g_2 = \frac{w_1x_1 + w_2x_2 + w_3x_3}{w_1 + w_2 + w_3}$$

Similarly, if G_3 is the c.g. of w_1, w_2, w_3, and w_4, and g_3 is its projection on the plane, then—

$$G_3g_3 = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4}{w_1 + w_2 + w_3 + w_4}, \text{ and so on}$$

and finally—

$$Gg \text{ or } \bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + w_5x_5}{w_1 + w_2 + w_3 + w_4 + w_5} \quad . \quad . \quad (2)$$

which may be written—

$$\bar{x} = \frac{\Sigma(wx)}{\Sigma(w)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where Σ stands for "the sum of all such terms as." If any of the points A, B, C, etc., are below the plane, their distances from the plane must be reckoned as negative.

Plane-moments.—The products w_1x_1, w_2x_2, w_3x_3, etc., are sometimes called *plane-moments* of the weights of the bodies about the plane considered. The plane-moment of a body about any given plane is then the weight of the body multiplied by the distance of its c.g. from that plane.

Then in words the relation (3) may be stated as follows: "The distance of the c.g. of several bodies (or of a body divided into parts) from any plane is equal to the algebraic sum of their several plane-moments about that plane, divided by the sum of their weights."

And since by (3), $\bar{x} \times \Sigma(w) = \Sigma(wx)$, we may state that the plane-moment of a number of weights (or forces) is equal to the sum of their several plane-moments.

This statement extends to plane-moments the statement in Art. 90, that the moment of the sum of several forces about any *point* is equal to the sum of the moments of the forces about that point.

It should be remembered that a horizontal plane was chosen for convenience only, and that the formulæ (2) and (3) hold good for distances from *any* plane.

114. Distance of the c.g. of an Area or Lamina from a Line in its Plane.

This is a particular case of the problem of the last article. Suppose the points A, B, C, D, and E in the last article and Fig. 116 all lie in one plane perpendicular to the horizontal plane, from which their distances are x_1, x_2, x_3, x_4, and x_5 respectively. Then their projections *a*, *b*, *c*, *d*, and *e* on the horizontal plane all lie in a straight line, which is the intersection of the plane containing A, B, C, D, and E with the horizontal plane, viz. the line OM in Fig. 117.

Thus, if x_1, x_2, x_3, etc., be the distances of the centres of gravity of several bodies all in the same plane (or parts of a lamina) from a fixed line OM in this plane, then the distance of the c.g. of the bodies (or laminæ) from the line being $\bar{x}$—

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + \ldots, \text{etc.}}{w_1 + w_2 + w_3 + \ldots, \text{etc.}} = \frac{\Sigma(wx)}{\Sigma(w)} \quad (4)$$

This formula may be used to find the position of the c.g. of

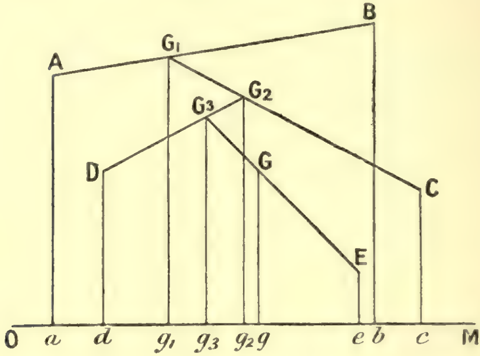


FIG. 117.

a lamina or area by finding its distance from two non-parallel fixed lines in its plane.

If the lamina is of irregular shape, as in Fig. 118, the distance of its c.g. from a line OM in its plane may be found approximately by dividing it into a number of narrow strips of equal width by lines parallel to OM, and taking the c.g. of each strip as being midway between the parallel boundary-lines. The weight of any strip being denoted by w—

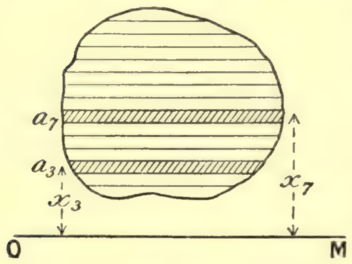


FIG. 118.

$$w = \text{volume of strip} \times D$$

where D = weight of unit volume of the material of the lamina, or—

$$w = \text{area of strip} \times \text{thickness of lamina} \times D$$

If the weight of the first, second, third, and fourth strips be w_1, w_2, w_3, and w_4 respectively, and so on, and their areas be a_1, a_2, a_3, and a_4 respectively, the lamina consisting of a material of uniform thickness t, then $w_1 = a_1t.D$, $w_2 = a_2t.D$, and

so on. And if $\bar{x}$ is the distance of the c.g. of the area from OM, then by equation (4)—

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots, \text{etc.}}{w_1 + w_2 + w_3 + \ldots, \text{etc.}}$$

$$= \frac{a_1tDx_1 + a_2tDx_2 + a_3tDx_3 + \ldots, \text{etc.}}{a_1tD + a_2tD + a_3tD + \ldots, \text{etc.}} \quad . \quad (5)$$

or, dividing numerator and denominator by the factor tD—

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots, \text{etc.}}{a_1 + a_2 + a_3 + a_4 + \ldots, \text{etc.}}$$

$$= \frac{\Sigma(ax)}{\Sigma(a)} \text{ or } \frac{\Sigma(ax)}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A = total area of the lamina, and Σ has the same meaning as in (3), Art. 113.

Similarly, the distance of the c.g. of the area A from another straight line may be found, and then the position of the c.g. is completely determined.

Thus in Fig. 119, if $\bar{x}$ is the distance of the c.g. of the lamina from OM, and $\bar{y}$ is its distance from ON, by drawing two lines, PR and QS, parallel to OM and ON and distant $\bar{x}$ and $\bar{y}$ from them respectively, the intersection G of the two lines gives the c.g. of the lamina or area.

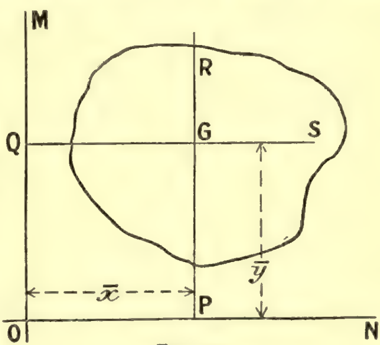


FIG. 119.

Moment of an Area.—The products a_1x_1, etc., may be called *moments of the areas* a_1, etc.

Regular Areas.—If a lamina consists of several parts, the centres of gravity of which are known, the division into thin strips adopted as an approximate method for irregular figures

is unnecessary. The distance $\bar{x}$ of the c.g. from any line OM is $\frac{\Sigma(ax)}{\Sigma(a)}$, or—

$$\bar{x} = \frac{\Sigma(\text{product of each area and distance of its c.g. from OM})}{\text{whole area}}$$

or—

$$\bar{x} = \frac{\Sigma(\text{plane mo. of each area about a plane perpend. to its own})}{\text{whole area}}$$

The product of an area and the distance of its c.g. from a line OM may be called the "line moment" of the area about OM, and we may write—

$$\bar{x} = \frac{\Sigma(\text{line moments of each part of an area})}{\text{whole area}}$$

For example, in Fig. 120 the area ABECD consists of a triangle, BEC, and a rectangle, ABCD, having a common side, BC. Let the height EF $= h$; let AD $= l$ and AB $= d$. Then the area ABCD $= d \times l$, and the area BEC $= \frac{1}{2} \times l \times h$, and if G_1 is the c.g. of the triangle BEC, and G_2 that of the rectangle ABCD, the distance $\bar{x}$ of the c.g. of the area ABECD from AD is found thus—

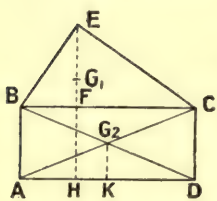


FIG. 120.

$$\bar{x} = \frac{(d\,.\,l) \times G_2K + \frac{1}{2}\,.\,l\,.\,h \times G_1H}{d\,.\,l + \frac{1}{2}\,.\,l\,.\,h} = \frac{d\,.\,l \times \frac{d}{2} + \frac{1}{2}lh(d + \frac{1}{3}h)}{l(d + \frac{1}{2}h)}$$

$$= \frac{d^2 + hd + \frac{1}{3}h^2}{2d + h}$$

115. Lamina with Part removed.—Suppose a lamina (Fig. 121) of area A has a portion of area a, removed. Let $\bar{x}$ = distance of c.g. G of A from a line OM in its plane; let x_1 be the distance of the c.g. of the part a from OM; and let x_2 be the distance of the c.g. of the remainder $(A - a)$ from OM.

$$\text{Then } \bar{x} = \frac{x_1a + x_2(A - a)}{A} \text{ (Art. 114)}$$

$$\bar{x} \,.\, A = x_1a + x_2(A - a)$$

$$\text{and } x_2 = \frac{\bar{x}A - x_1a}{A - a}$$

In this way we can find the distance of the c.g. of the part $A - a$ from OM, and similarly we can find the distance from

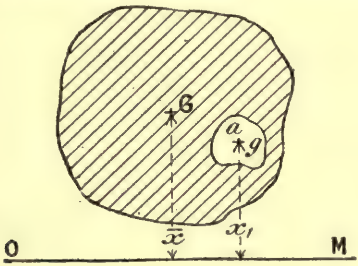


FIG. 121.

any other line in its plane, and so completely determine its position as in Art. 114. This method is applicable particularly to regular areas.

116. Solid with Part removed.—The method used in the last article to find the c.g. of part of a lamina is applicable to a solid of which part has been removed.

If in Fig. 122 A is a solid of weight W, and a portion B weighing w is removed, the distance of the c.g. of the remainder $(W - w)$ from any plane is x_2 where—

$$x_2 = \frac{\bar{x}W - x_1w}{W - w}$$

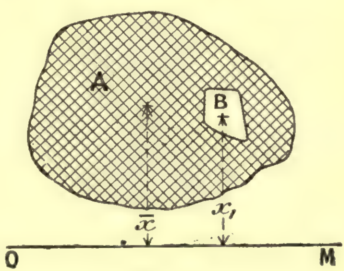


FIG. 122.

by (1) Art. 112 and the method of Art. 115, where $\bar{x}$ = distance of c.g. of A from the plane, and x_1 = distance of c.g. of B from the plane.

117. Centre of Gravity of a Circular Arc.—Let ABC (Fig. 123) be the arc, OA being the radius, equal to a units of length, and the length of arc ABC being l units. If B is the middle point of the arc, OB is an axis of symmetry, and the c.g. of the arc is in OB. Draw OM parallel to AC.

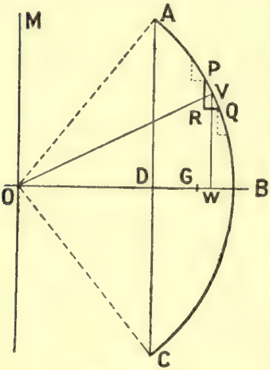


Fig. 123.

Let the arc be divided into a number of small portions, such as PQ, each of such small length as to be sensibly straight. Let the weight of the arc be w per unit length. The c.g. of a small portion PQ is at V, its mid-point. Draw VW parallel to OM, and join OV. Draw PR and QR parallel to OM and OB respectively.

Then, if $\bar{x}$ = distance of c.g. of arc from the line OM, as in Art. 114—

$$\bar{x} = \frac{\Sigma(\text{PQ} \times w \times \text{OW})}{\Sigma(\text{PQ}w)} = \frac{\Sigma(\text{PQ} \times \text{OW})}{\Sigma(\text{PQ})} = \frac{\Sigma(\text{PQ} \,.\, \text{OW})}{l}$$

Now, since OV, VW, and OW are respectively perpendicular to PQ, RQ, and PR, the triangles PQR and OVW are similar, and—

$$\frac{\text{PQ}}{\text{OV}} = \frac{\text{RP}}{\text{OW}}$$

$$\text{or PQ} \,.\, \text{OW} = \text{OV} \,.\, \text{RP} = a \,.\, \text{RP}$$

$$\text{hence } \Sigma(\text{PQ} \,.\, \text{OW}) = \Sigma(a \,.\, \text{RP}) = a\Sigma(\text{RP}) = a \times \text{AC}$$

and therefore—

$$\bar{x} = \frac{\Sigma(\text{PQ} \,.\, \text{OW})}{l} = \frac{a}{l} \,.\, \text{AC, or } \frac{\text{AC}}{l} \times a$$

The c.g. of the arc then lies in OB at a point G such that—

$$\text{OG} = \text{OB} \times \frac{\text{AC}}{l} \text{ or radius} \times \frac{\text{chord}}{\text{arc}}$$

or, if angle AOC = 2α, *i.e.* if angle AOB = α (radians)—

$$\text{OG} = a \times \frac{\text{AC}}{l} = a \times \frac{2\text{AD}}{l} = a \times \frac{2 \,.\, a \sin \alpha}{a \times 2\alpha} = a \,.\, \frac{\sin \alpha}{\alpha}$$

When the arc is very short, OG is very nearly equal to OB.

118. Centre of Gravity of Circular Sector and Segment.—Let the sector ABCO (Fig. 124) of a circle centred at O and of radius a, subtend an angle 2α at O. The sector may be divided into small parts, such as OPQ, by radial lines from O. Each such part is virtually triangular when PQ is so short as to be regarded as a straight line. The c.g. of the triangle OPQ is on the median OR, and $\frac{2}{3}a$ from O. Similarly, the centres of gravity of all the constituent triangles, such as PQO, lie on a concentric arc abc of radius $\frac{2}{3}a$ and subtending an angle 2α at O. The c.g. of the sector coincides with the c.g. of the arc abc, and is therefore in OB and at a distance $\frac{2}{3}a \cdot \frac{\sin \alpha}{\alpha}$ from O (Art. 117); *e.g.* the c.g. of a semicircular area of radius "a" is at a distance $\frac{2}{3}a \div \frac{\pi}{2}$ or $\frac{4a}{3\pi}$ from its straight boundary.

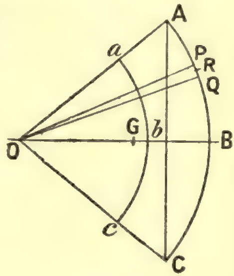


FIG. 124.

The c.g. of the segment cut off by any chord AC (Fig. 124) may be found by the principles of Art. 115, regarding the segment as the remainder of the sector ABCO when the triangle AOC is removed.

119. Centre of Gravity of a Zone of a Spherical Shell.—Let ABCD (Fig. 125) be a zone of a spherical shell of radius a and thickness t, and of uniform material which weighs w per unit volume. Let the length of axis HF be l. Divide the zone into a number of equal smaller zones, such as $abcd$, by planes perpendicular to the axis OE, so that each has an axial length h. Then the area of each small zone is the same,

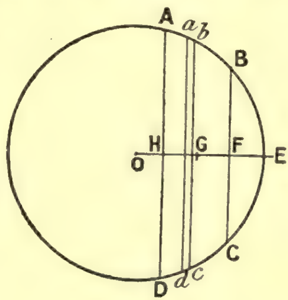


FIG. 125.

viz. $2\pi ah$, and the volume of each is then $2\pi ah \cdot t$, and each has its c.g. on the axis of symmetry OE, and midway between the bounding planes, such as *ad* and *bc*, if h is indefinitely short. Hence the c.g. of the zone coincides with that of a large number of small bodies each of weight $w \cdot 2\pi ah \cdot t$, having their centres of gravity uniformly spread along the line FH. Hence the c.g. is at G, the mid-point of the axis FH of the zone, or—

$$\text{OG} = \frac{\text{OF} + \text{OH}}{2}$$

e.g. the distance of the c.g. of a hemispherical shell from the plane of its rim is half the radius of the shell.

120. Centre of Gravity of a Sector of a Sphere.—Let OACB (Fig. 126) be a spherical sector of radius a. If the sector be divided into an indefinitely great number of equal small pyramids or cones having a common vertex O such that their bases together make up the base ACB of the sector, the c.g.'s of the equal pyramids will each be $\frac{3}{4}a$ from O, and will therefore be evenly spread over a portion *acb* (similar to the surface ACB) of a spherical surface centred at O and of radius $\frac{3}{4}a$. The c.g. of the sector then coincides with that of a zone, *acb*, of a thin spherical shell of radius $\frac{3}{4}a$, and is midway between c and the plane of the boundary circle *ab*, *i.e.* midway between d and c.

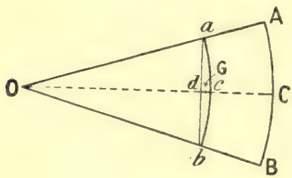


FIG. 126.

Solid Hemisphere.—The hemisphere is a particular case of a spherical sector, and its c.g. will coincide with that of a hemispherical shell of $\frac{3}{4}a$, where a is the radius of the solid hemisphere. This is a point on the axis of the solid hemisphere, and half of $\frac{3}{4}a$, or $\frac{3}{8}a$ from its base.

Example 1.—The base of a frustum of a cone is 10 inches diameter, and the smaller end is 6 inches diameter, the height being 8 inches. A co-axial cylindrical hole, 4 inches diameter, is bored through the frustum. Find the distance of the c.g. of the remaining solid from the plane of its base.

The solid of which the c g. is required is the remaining portion

of a cone, ABC (Fig. 127), when the upper cone, DBE, and a cylinder, FGKH, have been removed.

Since the cone diameter decreases 4 inches in a height of 8 inches—

The height BM $= 8 + 8 \times \frac{6}{4} = 20$ inches

and the c.g. of the cone ABC is $\frac{1}{4} \times 20$ inches $\Big\} = 5$ inches from AC

volume of cone ABC $= \pi \cdot (5)^2 \cdot \frac{20}{3} = \pi \cdot \frac{500}{3}$ cubic inches

distance from AC of c.g. of cylinder FGKH $\Big\} = \frac{8}{2} = 4$ inches

volume of cylinder FGKH $\Big\} = \pi \cdot 2^2 \cdot 8 = 32\pi$ cubic inches

volume of cone DBE $= \pi \cdot 3^2 \cdot \frac{12}{3} = 36\pi$ cubic inches

distance from AC of c.g. of cone DBE $\Big\} = 8 + \frac{12}{4} = 11$ inches

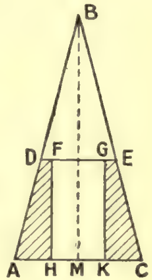


FIG. 127.

then volume of remaining frustum is—

$$\pi(\tfrac{500}{3} - 32 - 36) = \pi \cdot \tfrac{296}{3} \text{ cubic inches}$$

Let h = height of c.g. of this remainder from the base.

Then equating the plane-moments about the base of the three solids, BDE, FGKH, and the remainder of frustum, to the plane-moment of the whole cone (Art. 113) (and leaving out of both sides of equation the common factor weight per unit volume)—

$$\pi \cdot \tfrac{500}{3} \times 5 = \pi\{(32 \times 4) + (36 \times 11) + (\tfrac{296}{3} \times h)\}$$
$$833\dot{3} = 524 + \tfrac{296}{3}h$$
$$h = \tfrac{3}{296} \times 309\dot{3} = 3{\cdot}135 \text{ inches}$$

Example 2.—An I-section of a girder is made up of three rectangles, viz. two flanges having their long sides horizontal, and one web connecting them having its long side vertical. The top flange section is 6 inches by 1 inch, and that of the bottom flange is 12 inches by 2 inches. The web section is 8 inches deep and 1 inch broad. Find the height of the c.g. of the area of cross-section from the bottom of the lower flange.

Fig. 128 represents the section of the girder.

Let $\bar{x}$ = height of the c.g. of the whole section.

The height of the c.g. of BCDE is 1 inch above BE ;

„ „ FGHK is $2 + \frac{8}{2} = 6$ inches above BE ;

„ „ LMNP is $2 + 8 + \frac{1}{2} = 10{\cdot}5$ inches above BE.

Equating the sum of the moments of these three *areas* about A to the moment of the whole figure about A, we have—

$$(12 \times 2)1 + (8 \times 1)6 + (6 \times 1)10{\cdot}5 = \bar{x}\{(12\times 2) + (8\times 1) + (6\times 1)\}$$
$$24 + 48 + 63 = \bar{x}(24 + 8 + 6)$$
$$\bar{x} = \tfrac{135}{38} = 3{\cdot}55 \text{ inches}$$

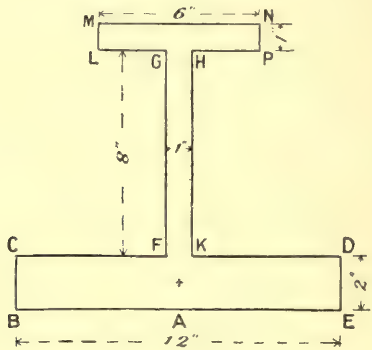


FIG. 128.

which is the distance of the c g. from the bottom of the lower flange.

Example 3.—Find the c.g. of a cast-iron eccentric consisting of a short cylinder 8 inches in diameter, having through it a cylindrical hole 2·5 inches diameter, the axis of the hole being parallel to that of the eccentric and 2 inches from it. State the distance of the c.g. of the eccentric from its centre.

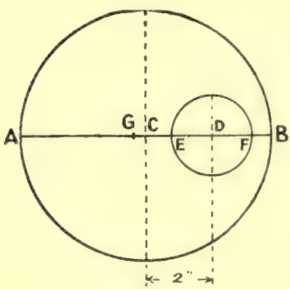


FIG. 129.

This is equivalent to finding the c.g. of the area of a circular lamina with a circular hole through it. In Fig. 129—

AB = 8 inches CD = 2 inches
EF = 2·5 inches

Let the distance of the c.g. from A be $\bar{x}$.

If the hole were filled with the same material as the remainder of the solid, the c.g. of the whole would be at C, its centre.

Equating moments of parts and the whole about A—

$$AC \times (\text{area of circle AB}) = (AD \times \text{area of circle EF}) + (\bar{x} \times \text{area of eccentric})$$

$$4 \times 64 = 6 \times 6{\cdot}25 + \bar{x}(64 - 6{\cdot}25)$$

$$\bar{x} = \frac{256 - 37{\cdot}5}{57{\cdot}75} = 3{\cdot}783$$

hence the distance of the c.g. from C is $4 - 3{\cdot}783$ or $0{\cdot}217$ inch.

Example 4.—A hemispherical shell of uniform material is 6 inches external radius and 1·5 inches thick. Find its c.g.

Let ABC (Fig. 130) be a solid hemisphere 12 inches diameter, from which a concentric solid hemisphere *abc*, 9 inches diameter, has been cut, leaving a hemispherical shell ACB*bca* 1·5 inches thick.

Let $\bar{x}$ = distance of its c.g. (which is on the axis of symmetry OC) from O.

Equating moments of volumes about O (*i.e.* omitting the factor of weight per unit volume)—

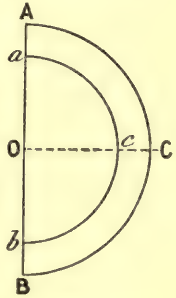


Fig. 130.

$$\left.\begin{array}{l}\text{Volume of solid}\\ \text{ABC} \times \tfrac{3}{8}\text{OC}\end{array}\right\} = (\text{volume of solid } acb \times \tfrac{3}{8}\text{O}c) + (\text{volume of shell} \times \bar{x})$$

$$\tfrac{2}{3}\pi 6^3 \times \tfrac{3}{8} \times 6 = \tfrac{2}{3}\pi \times (\tfrac{9}{2})^3 \times \tfrac{3}{8} \times \tfrac{9}{2} + \tfrac{2}{3}\pi\{6^3 - (\tfrac{9}{2})^3\}\bar{x}$$

from which $\bar{x} = 2{\cdot}66$ inches

The c.g. of the shell is on the axis and 2·66 inches from the centre of the surfaces.

Examples XV.

1. The front wheel of a bicycle is 30 inches diameter and weighs 4 lbs.; the back wheel is 28 inches diameter and weighs 7 lbs. The remaining parts of the bicycle weigh 16 lbs., and their c.g. is 18 inches forward of the back axle and 23 inches above the ground when the steering-wheel is locked in the plane of the back wheel. Find the c.g. of the whole bicycle; state its height above the ground and its distance in front of the back axle when the machine stands upright on level ground. The wheel centres are 42 inches horizontally apart.

2. A projectile consists of a hollow cylinder 6 inches external and 3 inches internal diameter, and a solid cone on a circular base 6 inches diameter, coinciding with one end of the cylinder. The axes of the cone and cylinder are in line; the length of the cylinder is 12 inches, and the

height of the cone is 8 inches. Find the distance of the c.g. of the projectile from its point.

3. A solid of uniform material consists of a cylinder 4 inches diameter and 10 inches long, with a hemispherical end, the circular face of which coincides with one end of the cylinder. The other end of the cylinder is pierced by a cylindrical hole, 2 inches diameter, extending to a depth of 7 inches along the cylinder and co-axial with it. Find the c.g. of the solid. How far is it from the flat end?

4. The profile of a crank (Fig. 131) consists of two semicircular ends, CED and AFB, of 8 inches and 12 inches radii respectively, centred at points P and O 3 feet apart, and joined by straight lines AC and BD. The crank is of uniform thickness, perpendicular to the figure, and is pierced by a hole 10 inches diameter, centred at O. Find the distance of the c.g of the crank from the axis O.

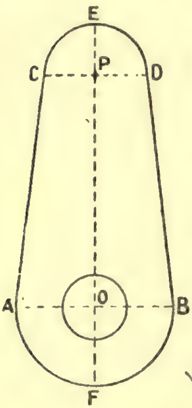


Fig. 131.

5. Find the c.g. of a **T** girder section, the height over all being 8 inches, and the greatest width 6 inches, the metal being $\frac{7}{8}$ inch thick in the vertical web, and 1 inch thick in the horizontal flange.

6. An **I**-section girder consists of a top flange 6 inches by 1 inch, a bottom flange 10 inches by 1·75 inches, connected by a web 10 inches by 1·15 inches. Find the height of the c.g. of the section from the lowest edge.

7. A circular lamina 4 inches diameter has two circular holes cut out of it, one 1·5 inches and the other 1 inch diameter with their centres 1 inch and 1·25 inches respectively from the centre of the lamina, and situated on diameters mutually perpendicular. Find the c.g. of the remainder of the lamina.

8. A balance weight in the form of a segment of a circle fits inside the rim of a wheel, the internal diameter of which is 3 feet. If the segment subtends an angle of 60° at the centre of the wheel, find the distance of its c.g. from the axis.

9. If two intersecting tangents are drawn from the extremities of a quadrant of a circle 4 feet diameter, find the distance of the c.g. of the area enclosed between the tangents and the arc, from either tangent.

10. A balance weight of a crescent shape fits inside the rim of a wheel of 6 feet internal diameter, and subtends an angle of 60° at its centre. The inner surface of the weight is curved to twice the radius of the outer surface, *i.e.* the centre from which its profile is struck is on the circumference of the inside of the wheel. The weight being of uniform thickness perpendicular to the plane of the wheel, find the distance of its c.g. from the axis of the wheel.

N.B.—The profile is equivalent to the sector of a circle plus two triangles minus a sector of a larger circle.

CHAPTER VIII

CENTRE OF GRAVITY: PROPERTIES AND APPLICATIONS

121. Properties of the Centre of Gravity.—Since the resultant force of gravity always acts through the centre of gravity, the weight of the various parts of a rigid body may be looked upon as statically equivalent to a single force equal to their arithmetic sum acting vertically through the centre of gravity of the body. Such a single force will produce the same reactions on the body from its supports; will have the same moment about any point (Art. 90); may be replaced by the same statically equivalent forces or components; and requires the same equilibrants, as the several forces which are the weights of the parts. Hence, if a body be supported by being suspended by a single thread or string, the c.g. of the body is in the same vertical line as that thread or string. If the same body is suspended again from a different point in itself, the c.g. is also in the second vertical line of suspension. If the two lines can be drawn on or in the body, the c.g., which must lie at their intersection, can thus be found experimentally. For example, the c.g. of a lamina may be found by suspending it from two different points in its perimeter, first from one and then from the other, so that its plane is in both cases vertical, and marking upon it two straight lines which are continuations of the suspension thread in the two positions.

Fig. 132 shows G, the c.g. of a lamina PQRS, lying in both the lines of suspension PR and QS from P and Q respectively. The tension of the cord acts vertically upwards on the lamina, and is equal in magnitude to the vertical downward force of

the weight of the lamina acting through G. The tension can only balance the weight if it acts through G, for in order that two forces may keep a body in equilibrium, they must be con-

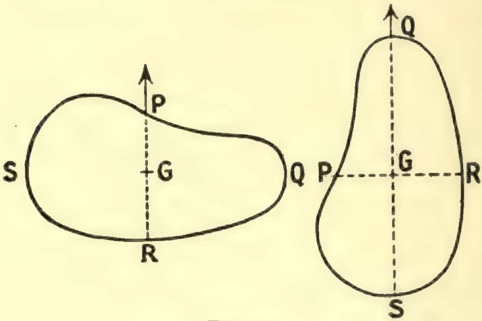


FIG. 132.

current, equal, and opposite, and therefore in the same straight line.

A "plumb line," consisting of a heavy weight hanging from a thin flexible string, serves as a convenient method of obtaining a vertical line.

122. Centre of Gravity of a Distributed Load.—If a load is uniformly distributed over the whole span of a beam, the centre of gravity of the load is at mid-span, and the reactions of the supports of the beam are the same as would be produced by the whole load concentrated at the middle of the beam. Thus, if in Fig. 133 a beam of 20-feet span carries a load of 3 tons per foot of span (including the weight of the beam) uniformly spread over its length, the reactions at the supports A and B are each the same as would be produced by a load of 60 tons acting at C, the middle section of the beam, viz. 30 tons at each support. Next suppose the load on a beam is distributed, not evenly, but in some known manner. Suppose the load per foot of span at various points to be

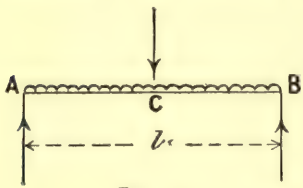


FIG. 133.

shown by the height of a curve ACDEB (Fig. 134). The load may be supposed to be piled on the beam, so that the curve ACDEB is its profile, and so that the space occupied is of constant thickness in a direction perpendicular to the plane of the figure. Then the c.g. of the load is at the c.g. G of

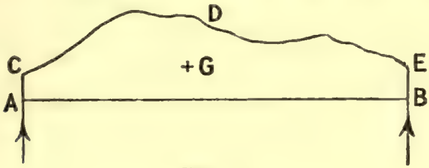


FIG. 134.

the area of a section such as ACDEB in Fig. 134, taken halfway through the constant thickness. The reactions of the supports are the same as if the whole load were concentrated at the point G. The whole load is equal to the length of the beam multiplied by the mean load per unit length, which is represented by the mean ordinate of the curve ACDEB, *i.e.* a length equal to the area ACDEB divided by AB.

Example.—As a particular case of a beam carrying a distributed load not evenly spread, take a beam of 20-feet span carrying a load the intensity of which is 5 tons per foot run at one end, and varying uniformly to 3 tons per foot at the other. Fig. 135 represents the distribution of load. Find the reactions at A and B.

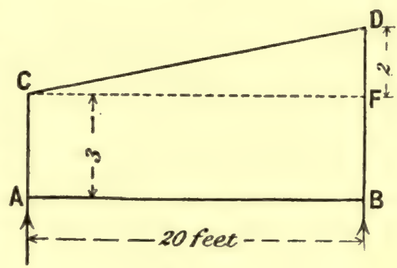


FIG. 135.

The total load = 20 × mean load per foot = $20 \times \frac{5+3}{2}$ = 80 tons

Let $\bar{x}$ be the distance of the c.g. of area ABCD from BD.

$$\bar{x}(\text{area ACFB} + \text{area CDF}) = (10 \times \text{area ACFB}) + (\tfrac{20}{3} \times \text{area CDF})$$

$$\bar{x}(3 \times 20 + \tfrac{1}{2} \cdot 20 \times 2) = (10 \times 20 \times 3) + \tfrac{20}{3} \times \tfrac{20}{2} \times 2$$

$$\bar{x} = \frac{600 + 133\cdot\dot{3}}{80} = 9\cdot1\dot{6} \text{ feet}$$

and distance of c.g. from AC = $20 - 9\cdot1\dot{6} = 10\cdot8\dot{3}$ feet

If R_A and R_B be the reactions at A and B respectively, equating opposite moments about B of all the forces on the beam—

$$R_A \times 20 = 80 \times 9{\cdot}1\dot{6}$$

$$R_A = 80 \times \frac{9{\cdot}1\dot{6}}{20} = 36{\cdot}\dot{6} \text{ tons}$$

$$R_B = 80 - 36{\cdot}\dot{6} = 43{\cdot}\dot{3} \text{ tons}$$

123. Body resting upon a Plane Surface.—As in the case of a suspended body, the resultant of all the supporting forces must pass vertically through the c.g. of the body in order to balance the resultant gravitational forces in that straight line. The vertical line through the c.g. must then cut the surface, within the area of the extreme outer polygon or curved figure which can be formed by joining all the points of contact with the plane by straight lines. If the vertical line through the c.g. fall on the perimeter of this polygon the solid is on the point of overturning, and if it falls outside that area the solid will topple over unless supported in some other way. This is sometimes expressed by saying

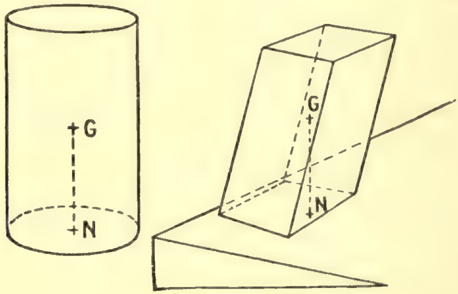


FIG. 136.

that a body can only remain at rest on a plane surface if the vertical line through the c.g. falls within the base. From what is stated above, the term "*base*" has a particular meaning, and does not signify only areas of actual contact; *e.g.* in Fig. 136 are two solids in equilibrium, with GN, the vertical line through G, the c.g., falling within the area of contact;

but in Fig. 137 a solid is shown in which the vertical through the c.g. falls outside the area of contact when the solid rests upright with one end on a horizontal plane. If, however, it falls within the extreme area ABC, the solid can rest in equilibrium on a plane.

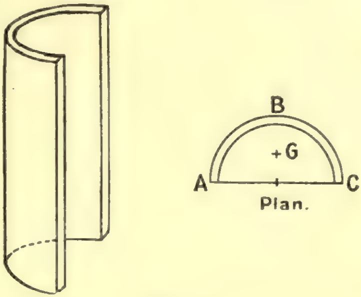


FIG. 137.

Two cases in which equilibrium is impossible are shown in Fig. 138, the condition stated above being violated. The first is that of a high cylinder on an inclined plane, and the second

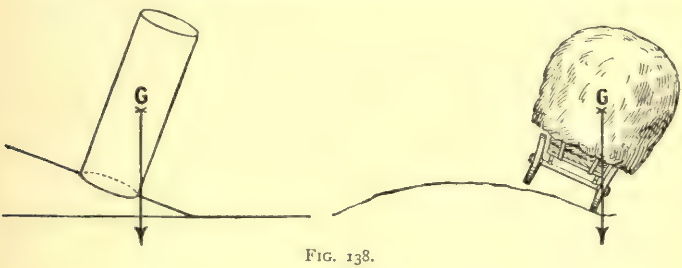


FIG. 138.

that of a waggon-load of produce on the side of a high crowned road. It will be noticed that a body subjected to tilting will topple over with less inclination or more, according as its c.g. is high or low.

Example.—What is the greatest length which a right cylinder of 8 inches diameter may have in order that it may rest with one end on a plane inclined 20° to the horizontal?

The limiting height will be reached when the c.g. falls vertically over the circumference of the base, *i.e.* when G (Fig. 139) is

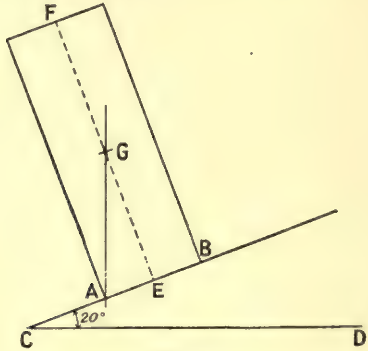


Fig. 139.

vertically above A. Then, G being the mid-point of the axis EF, the half-length of cylinder—

$$GE = AE \cot A\hat{G}E = AE \cot ACD$$
$$\text{or } GE = AE \cot 20^\circ = 4 \times 2{\cdot}7475 = 10{\cdot}99 \text{ inches}$$

The length of cylinder is therefore $2 \times 10{\cdot}99 = 21{\cdot}98$ inches.

124. Stable, Unstable, and Neutral Equilibrium.—A body is said to be in *stable* equilibrium when, if slightly disturbed from its position, the forces acting upon it tend to cause it to return to that position.

If, on the other hand, the forces acting upon it after a slight displacement tend to make it go further from its former position, the equilibrium is said to be *unstable*.

If, after a slight displacement, the forces acting upon the body form a system in equilibrium, the body tends neither to return to its former position nor to recede further from it, and the equilibrium is said to be *neutral*.

A few cases of equilibrium of various kinds will now be considered, and the conditions making for stability or otherwise.

125. Solid Hemisphere resting on a Horizontal Plane.—If a solid hemisphere, ABN (Fig. 140), rests on a

horizontal plane, and receives a small tilt, say through an angle θ, the c.g., situated at G, $\frac{3}{8}$ of ON from O and in the radius ON, takes up the position shown on the right hand of

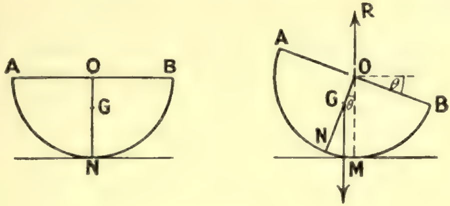


FIG. 140.

the figure. The forces acting instantaneously on the solid are then—(1) the weight vertically through G, and (2) the reaction R in the line MO vertically through M (the new point of contact between hemisphere and plane) and normal to the curved surface. These two forces form a "righting couple," and evidently tend to rotate the solid into its original position. Hence the position shown on the left is one of *stable* equilibrium. Note that G lies *below* O.

126. Solid with a Hemispherical End resting on a Horizontal Plane.—Suppose a solid consisting of, say,

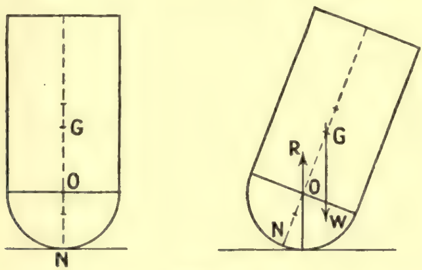


FIG. 141.

a cylinder with a hemispherical base, the whole being of homogeneous material, rests on a plane, and the c.g. G (Fig. 141) falls within the cylinder, *i.e.* beyond the centre O of the hemispherical end reckoned from N, where the axis cuts the

curved surface. On the left of Fig. 141 the solid is shown in a vertical position of equilibrium. Now suppose it to receive a slight angular displacement, as on the right side of the figure. The weight W, acting vertically downwards through G, along with the vertical reaction R of the plane, forms a system, the tendency of which is to move the body so that G moves, not towards its former position, but away from it. The weight acting vertically through G and the reaction of the plane acting vertically through O form an "upsetting couple" instead of a "righting couple." Hence the position on the left of Fig. 141 is one of *unstable* equilibrium. Note that in this case G falls above O. If the upper part of the body were so small that G is below O, the equilibrium would be stable, as in the case of the hemisphere above (Art. 125). The lower G is, the greater is the righting couple (or the greater the stability) for a given angular disturbance of the body. While in the case of instability, the higher G is, the greater is the upsetting couple or the greater the instability, and we have seen that such a solid is stable or unstable according as G falls below or above O.

127. Critical Case of Equilibrium neutral.—If G coincides with the centre of the hemisphere (Art. 126), the equilibrium is neither stable nor unstable, but neutral. Suppose the cylinder is shortened so that G, the c.g. of the whole solid, falls on O, the centre of the hemisphere. Then if the solid receives a slight angular displacement, as in the right side of Fig. 142, the reaction R of the plane acts vertically upwards through O, the centre of the hemisphere (being normal to the surface at the point of contact), and the resultant force of gravity acts vertically downward through the same point. In this case the two vertical forces balance, and there is no couple formed, and no tendency to rotate the body towards or away from its former position. Hence the equilibrium is neutral.

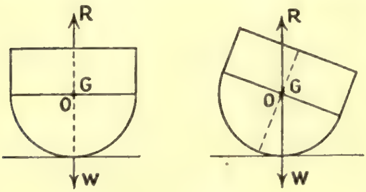


Fig. 142.

In each of the above instances the equilibrium as regards angular displacements is the same whatever the direction of the displacement. As further examples of neutral equilibrium, a sphere or cylinder of uniform material resting on a horizontal plane may be taken. The sphere is in neutral equilibrium with regard to angular displacements in any direction, but the horizontal cylinder (Fig. 143) is only in neutral equilibrium as regards its rolling displacements; in other directions its equilibrium is stable.

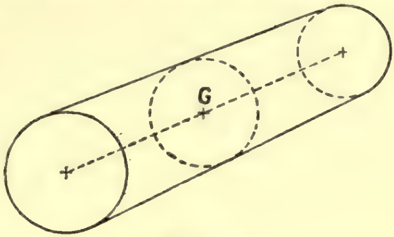


FIG. 143.

Example.—A cone and a hemisphere of the same homogeneous material have a circular face of 1 foot radius in common. Find for what height of the cone the equilibrium of the compound solid will be neutral when resting with the hemispherical surface on a horizontal plane.

The equilibrium will be neutral when the c.g. of the solid is at the centre of the hemisphere, *i.e.* at the centre O (Fig. 144) of their common face.

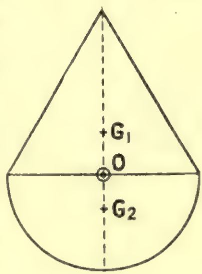


FIG. 144.

Let h be the height of the cone in feet. Then its c.g. G_1 is $\frac{1}{4}h$ from O, and its volume is $\frac{1}{3}h \times \frac{\pi}{4} \times 2^2 = \frac{1}{3}\pi h$ cubic feet.

The c.g. G_2, of the hemisphere is at $\frac{3}{8}$ foot from O, and its volume is $\frac{2}{3}\pi$ cubic feet. Then—

$$\frac{G_1O}{G_2O} = \frac{\frac{1}{4}h}{\frac{3}{8}} = \frac{\text{weight of hemisphere}}{\text{weight of cone}} = \frac{\frac{2}{3}\pi}{\frac{1}{3}\pi h}$$

$$\text{and } \tfrac{2}{3}h = \frac{2}{h}$$

$$h^2 = 3$$

$$h = \sqrt{3} = 1\cdot732 \text{ feet}$$

If h is greater than $\sqrt{3}$ feet the equilibrium is unstable, and if it is less than $\sqrt{3}$ feet the equilibrium is stable.

128. In the case of bodies resting on plane surfaces and having more than one point of contact, the equilibrium will be stable if the c.g. falls *within* the area of the *base*, giving the word the meaning attached to it in Art. 123 for small angular displacements in any direction. If the c.g. falls on the perimeter of the base, the equilibrium will be unstable for displacements which carry the c.g. outside the space vertically above the "base."

The attraction of the earth tends to pull the c.g. of a body into the lowest possible position; hence, speaking generally, the lower the c.g. of a body the greater is its stability, and the higher the c.g. the less stable is it.

In the case of a body capable of turning freely about a horizontal axis, the only position of stable equilibrium will be that in which the c.g. is vertically below the axis. When it

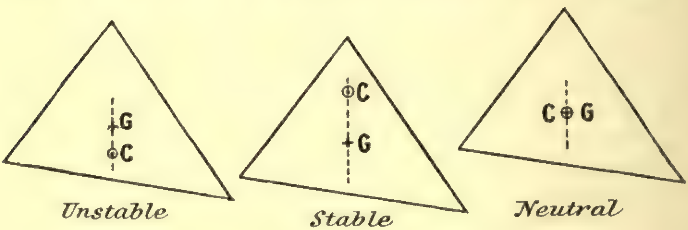


FIG. 145.

is vertically above, the equilibrium is unstable, and unless the c.g. is in the axis there are only two positions of equilibrium. If the c.g. is in the axis, the body can rest in neutral equilibrium in any position.

Fig. 145 represents a triangular plate mounted on a horizontal axis, C; it is in unstable, stable, or neutral equilibrium according as the axis C is below, above, or through G, the c.g. of the plate.

129. Work done in lifting a Body.—When a body is lifted, it frequently happens that different parts of it are lifted through different distances, *e.g.* when a hanging chain is wound up, when a rigid body is tilted, or when water is raised from one vessel to a higher one. The total work done in lifting the

body can be reckoned as follows: Let w_1, w_2, w_3, w_4, etc., be the weights of the various parts of the body, which is supposed divided into any number of parts, either large or small, but such that the whole of one part has exactly the same displacement (this condition will in many cases involve division into indefinitely small parts). Let the parts w_1, w_2, w_3, etc., be at heights x_1, x_2, x_3, etc., respectively above some fixed horizontal plane; if the parts are not indefinitely small, the distances x_1, x_2, x_3, etc., refer to the heights of their centres of gravity. Then the distance $\bar{x}$ of the c.g. from the plane is $\frac{\Sigma(wx)}{\Sigma(w)}$ (Art. 113). After the body has been lifted, let x_1', x_2', x_3', etc., be the respective heights above the fixed plane of the parts weighing w_1, w_2, w_3, etc. Then the distance $\bar{x}'$ of the c.g. above the plane is $\frac{\Sigma(wx')}{\Sigma(w)}$ (Art. 113).

The work done in moving the part weighing w_1 is equal to the weight w_1 multiplied by the distance $(x_1' - x_1)$ through which it is lifted; *i.e.* the work is $w_1(x_1' - x_1)$ units.

Similarly, the work done in lifting the part weighing w_2 is $w_2(x_2' - x_2)$. Hence the total work done is—

$$w_1(x_1' - x_1) + w_2(x_2' - x_2) + w_3(x_3' - x_3) +, \text{etc.}$$

which is equal to—

$$(w_1x_1' + w_2x_2' + w_3x_3' +, \text{etc.}) - (w_1x_1 + w_2x_2 + w_3x_3 +, \text{etc.})$$
$$\text{or } \Sigma(wx') - \Sigma(wx)$$
$$\text{But } \Sigma(wx') = \bar{x}'\Sigma(w) \text{ and } \Sigma(wx) = \bar{x}\Sigma(w)$$
$$\text{therefore the work done } = \bar{x}'\Sigma(w) - \bar{x}\Sigma(w)$$
$$= (\bar{x}' - \bar{x})\Sigma(w)$$

The first factor, $\bar{x}' - \bar{x}$, is the distance through which the c.g. of the several weights has been raised, and the second factor, $\Sigma(w)$, is the total weight of all the parts. Hence the total work done in lifting a body is equal to the weight of the body multiplied by the vertical distance through which its c.g. has been raised.

Example 1.—A rectangular tank, 3 feet long, 2 feet wide, and 1·5 feet deep, is filled from a cylindrical tank of 24 square feet horizontal cross-sectional area. The level of water, before filling begins, stands 20 feet below the bottom of the rectangular tank. How much work is required to fill the tank, the weight of 1 cubic foot of water being 62·5 lbs.?

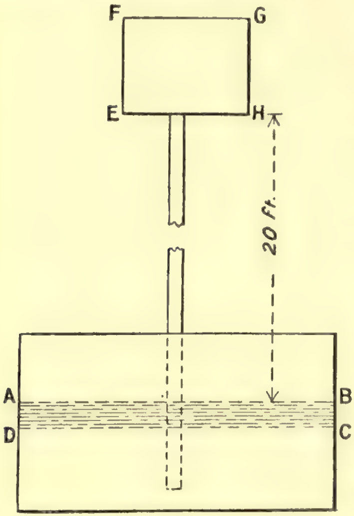


FIG. 146.

The water to be lifted is $3 \times 2 \times 1·5$ or 9 cubic feet, hence the level in the lower tank will be lowered by $\frac{9}{24}$ or $\frac{3}{8}$ of a foot, *i.e.* by a length BC on Fig. 146. The 9 cubic feet of water lifted occupies first the position ABCD, and then fills the tank EFGH. In the former position its c.g. is $\frac{1}{2}$BC or $\frac{3}{16}$ foot below the level AB, and in the latter position its c.g. is $\frac{1}{2}$GH or $\frac{3}{4}$ foot above the level EH. Hence the c.g. is lifted $(\frac{3}{16} + 20 + \frac{3}{4})$ feet, *i.e.* $20\frac{15}{16}$ feet, or 20·9375 feet.

The weight of the 9 cubic feet of water lifted is $9 \times 62·5 = 562·5$ lbs.

Hence the work done is $562·5 \times 20·9375 = 11,770$ foot-lbs.

Example 2.—Find the work in foot-pounds necessary to upset a solid right circular cylinder 3 feet diameter and 7 feet high, weighing half a ton, which is resting on one end on a horizontal plane.

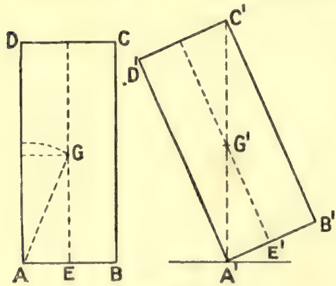


FIG. 147.

Suppose the cylinder (Fig. 147) to turn about a point A on the circumference of the base. Then G, the c.g. of the cylinder, which was formerly 3·5 feet above the level of the horizontal plane, is raised to a position G′, *i.e.* to a height A′G′ above the horizontal plane before the cylinder is overthrown.

The distance the c.g. is lifted is then A′G′ − EG—

$$A'G' = \sqrt{(AE^2 + EG^2)} = \sqrt{(1{\cdot}5^2 + 3{\cdot}5^2)} = 3{\cdot}807 \text{ feet}$$

The c.g. is lifted $3{\cdot}807 - 3{\cdot}5 = 0{\cdot}307$ foot

and the work done is $1120 \times 0{\cdot}307 = 344$ foot-lbs.

Example 3.—A chain 600 feet long hangs vertically; its weight at the top end is 12 lbs. per foot, and at the bottom end 9 lbs. per foot, the weight per foot varying uniformly from top to bottom. Find the work necessary to wind up the chain.

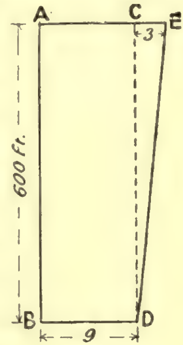


FIG. 148.

It is first necessary to find the total weight of the chain and the position of its c.g. The material of the chain may be considered to be spread laterally into a sheet of uniform thickness, the length remaining unchanged. The width of the sheet will then be proportional to the weight per foot of length; the total weight, and the height of the c.g. of the chain, will not be altered in such a case.

The depth of the c.g. below the highest point (A) of the chain (Fig. 148) will be the same as that of a figure made up of a rectangle, ACDB, 600 feet long and 9 (feet or other units) broad, and a right-angled triangle, CED, having sides about the right angle at C of (CD) 600 feet and (CE) 3 units.

The depth will be—

$$\frac{(600 \times 9 \times 300) + (\frac{1}{2} \times 600 \times 3 \times \frac{600}{3})}{(600 \times 9) + (\frac{1}{2} \times 600 \times 3)} \quad \text{(Art. 114)}$$

which is equal to 285·7 feet.

The total weight of the chain will be the same as if it were 600 feet long and of uniform weight $\frac{12 + 9}{2}$ or 10·5 lbs. per foot, viz. $600 \times 10{\cdot}5 = 6300$ lbs.

Hence the work done in raising the chain all to the level A is—

$$6300 \times 285{\cdot}7 = 1{,}800{,}000 \text{ foot-lbs.}$$

130. Force acting on a Rigid Body rotating uniformly about a Fixed Axis.

Let Fig. 149 represent a cross-section of a rigid body of weight W rotating about a fixed axis, O, perpendicular to the figure. For simplicity the body will be supposed symmetrical

about the plane of the figure, which therefore contains G, the c.g. of the body. In the position shown, let w_1 be the weight of a very small portion of the body (cut parallel to the axis) situated at a distance r from O. Let ω be the uniform angular velocity of the body about the axis O. Then the force acting upon the small portion of weight w_1 in order to make it rotate about O is $\frac{w_1}{g}\omega^2 r$, directed towards O (Art. 63), and it evidently acts at the middle of the length of the portion, *i.e.* in the plane of the figure. Resolving this force in any two perpendicular directions, XO and YO, the components in these two directions are $\frac{w_1}{g}\omega^2 r \cos\theta$ and $\frac{w_1}{g}\omega^2 r \sin\theta$ respectively, where θ is the angle which AO makes with OX.

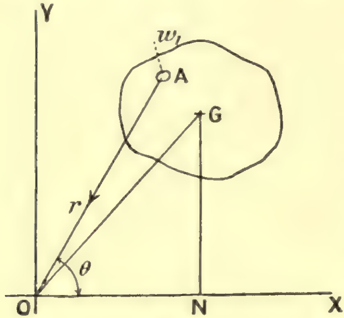


FIG. 149.

These may be written $\frac{w_1}{g} \cdot \omega^2 \cdot x$ and $\frac{w_1}{g}\omega^2 \cdot y$ respectively, where x represents $r \cos\theta$ and y represents $r \sin\theta$, the projections of r on OX and OY respectively.

Adding the components in the direction XO of the centripetal forces acting in the plane of the figure upon all such portions making up the entire solid, the total component—

$$F_X = \Sigma\left(\frac{w}{g}\omega^2 x\right) = \frac{\omega^2}{g}\Sigma(wx) = \frac{\omega^2}{g}\bar{x}\Sigma(w) = \frac{W}{g} \cdot \omega^2 \cdot \bar{x}$$

and the total component force in the direction YO is—

$$F_Y = \Sigma\left(\frac{w}{g}\omega^2 y\right) = \frac{\omega^2}{g} \cdot \Sigma(wy) = \frac{W}{g} \cdot \omega^2 \cdot \bar{y}$$

where $\bar{x}$ and $\bar{y}$ are the distances of G, the c.g. of the solid (which is in the plane of the figure), from OY and OX respectively.

Hence the resultant force P acting on the solid towards O is—

$$P = \sqrt{(F_X{}^2 + F_Y{}^2)} = \frac{\omega^2}{g} \cdot W \cdot \sqrt{(\bar{x}^2 + \bar{y}^2)} = \frac{W}{g} \cdot \omega^2 \cdot R$$

where $R = \sqrt{\bar{x}^2 + \bar{y}^2}$, the distance of the c.g. from the axis O.

Hence the resultant force acting on the body is of the same magnitude as the centripetal force $\left(\frac{W}{g}\omega^2 R\right)$ which must act on a weight W concentrated at a radius R from O in order that it may rotate uniformly at an angular velocity ω. Further, the tangent of the angle which P makes with XO is $\frac{F_Y}{F_X}$ (Art. 75), which is equal to $\frac{\bar{y}}{\bar{x}}$, or $\frac{GN}{ON}$, where GN is perpendicular to OX. Hence the force P acts in the line GO, and therefore the resultant force P acting on the rotating body is in all respects identical with that which would be required to make an equal weight, W, rotate with the same angular velocity about O if that weight were concentrated (as a particle) at G, the c.g. of the body.

It immediately follows, from the third law of motion, that the *centrifugal* force exerted *by* the rotating body on its constraints is also of this same magnitude and of opposite direction in the same straight line.

Example.—Find the force exerted on the axis by a thin uniform rod 5 feet long and weighing 9 lbs., making 30 revolutions per minute about an axis perpendicular to its length.

FIG. 150.

The distance from the axis O to G, the c g. of the rod (Fig. 150), is 2·5 feet, the c.g. being midway between the ends. The angular velocity of the rod is $\frac{30 \times 2\pi}{60} = \pi$ radians per second. The centrifugal pull on O is the same as that of a weight of 9 lbs. concentrated at 2·5 feet from the axis and describing about O, π radians per second, which is—

$$\frac{9}{32 \cdot 2} \times \pi^2 \times 2 \cdot 5 = 6 \cdot 89 \text{ lbs.}$$

131. Theorems of Guldinus or Pappus.—(*a*) The area of the surface of revolution swept out by any plane curve revolving about a given axis in its plane is equal to the length of the curve multiplied by the length of the path of its c.g. in describing a circle about the axis. Suppose the curve ABC (Fig. 151) revolves about the axis OO′, thereby generating a surface of revolution of which OO′ is the axis. Let S be the length of the curve, and suppose it to be divided into a large number of small parts, s_1, s_2, s_3, etc., each of such short length that if drawn straight the shape of the curve is not appreciably altered. Let the distances of the parts s_1, s_2, s_3, etc., from the axis be x_1, x_2, x_3, etc.; and let G, the c.g. of the curve which is in the plane of the figure, *i.e.* the plane of the curve, be distant $\bar{x}$ from the axis OO′. The portion s_1 generates a surface the length of which is $2\pi x_1$ and the breadth s_1; hence the area is $2\pi x_1 s_1$. Similarly, the portion s_2 generates an area $2\pi x_2 . s_2$, and the whole area is the sum—

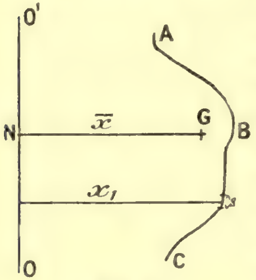


FIG. 151.

$$2\pi x_1 s_1 + 2\pi x_2 s_2 + 2\pi x_3 s_3 + \text{, etc., or } 2\pi\Sigma(xs)$$

If the portions s_1, s_2, s_3, etc., are of finite length, this result is only an approximation; but if we understand $\Sigma(xs)$ to represent the limiting value of such a sum, when the length of each part is reduced indefinitely, the result is not a mere approximation.

Now, since $\Sigma(xs) = \bar{x} \times \Sigma(s) = \bar{x} \times \text{S}$, the whole area of the surface of revolution is $2\pi\bar{x} . \text{S}$, of which $2\pi\bar{x}$ is the length of the path of the c.g. of the curve in describing a circle about OO′, and S is the length of the curve.

(*b*) The volume of a solid of revolution generated by the revolution of a plane area about an axis in its plane is equal to the enclosed revolving area multiplied by the length of the path of the c.g. of that area in describing a complete circle about the axis.

Suppose that the area ABC (Fig. 152) revolves about the axis OO′, thereby generating a solid of revolution of which

OO′ is an axis (and which is enclosed by the surface generated by the perimeter ABC).

Let the area of the plane figure ABC be denoted by A, and let it be divided into a large number of indefinitely small parts a_1, a_2, a_3, etc., situated at distances x_1, x_2, x_3, etc., from the axis OO′.

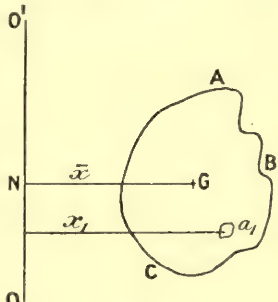


FIG. 152.

The area a_1, in revolving about OO′, generates a solid ring which has a cross-section a_1 and a length $2\pi x_1$, and therefore its volume is $2\pi x_1 a_1$. Similarly, the volume swept out by the area a_2 is $2\pi x_2 a_2$, and so on. The whole volume swept out by the area A is the limiting value of the sum of the small quantities—

$$2\pi x_1 a_1 + 2\pi x_2 a_2 + 2\pi x_3 a_3 + \text{, etc.,}$$
$$\text{or } 2\pi(a_1 x_1 + a_2 x_2 + a_3 x_3 + \text{, etc.,}) \text{ or } 2\pi\Sigma(ax)$$

And since $\Sigma(ax) = \bar{x}\Sigma(a) = \bar{x} \cdot A$ (Art. 114 (6)), the whole volume is $2\pi\bar{x} \cdot A$, of which $2\pi\bar{x}$ is the length of the path of the c.g. of the area in describing a circle about the axis OO′, and A is the area.

Example.—A groove of semicircular section 1·25 inches radius is cut in a cylinder 8 inches diameter. Find (*a*) the area of the curved surface of the groove, and (*b*) the volume of material removed.

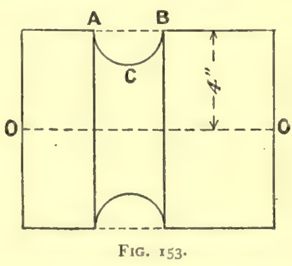


FIG. 153.

(*a*) The distance of the c.g. of the semicircular arc ABC (Fig. 153) from AB is $\left(1{\cdot}25 \times \frac{2}{\pi}\right)$ or $\frac{2{\cdot}5}{\pi}$ inches. Therefore the distance of the c.g. of the arc from the axis OO′ is $\left(4 - \frac{2{\cdot}5}{\pi}\right)$ inches. The length of path of this point in making one complete circuit about

OO′ is $2\pi\left(4 - \frac{2\cdot5}{\pi}\right) = (8\pi - 5)$ inches. The length of arc ABC is $1\cdot25\pi$ inches, hence the area of the surface of the semicircular groove is—

$$
\begin{aligned}
1\cdot25\pi(8\pi - 5) \text{ square inches} &= 10\pi^2 - 6\cdot25\pi \\
&= 98\cdot7 - 19\cdot6 \\
&= 79\cdot1 \text{ square inches}
\end{aligned}
$$

(*b*) The distance of the c.g. of the area ABC from AB is $\frac{4}{3\pi} \times 1\cdot25 = 0\cdot530$ inch, and therefore the distance of the c.g. from OO′ is $4 - 0\cdot53 = 3\cdot47$ inches.

The length of path of this point in making one complete circuit about OO′ is $2\pi \times 3\cdot47 = 21\cdot8$ inches. The area of the semicircle is $\frac{1}{2}(1\cdot25)^2\pi = 2\cdot453$ square inches, hence the volume of the material removed from the groove is—

$$21\cdot8 \times 2\cdot453 = 53\cdot5 \text{ cubic inches}$$

132. Height of the c.g. of a Symmetrical Body, such as a Carriage, Bicycle, or Locomotive.—It was stated in Art. 121 that the c.g. of some bodies might conveniently be found experimentally by suspending the bodies from two different points in them alternately. This is not always convenient, and a method suitable for some other bodies will now be explained by reference to a particular instance. The c.g. of a bicycle (which is generally nearly symmetrical about a vertical plane through both wheels) may be determined by first finding the vertical downward pressure exerted by each wheel on the level ground, and then by finding the vertical pressures when one wheel stands at a measured height above the other one.

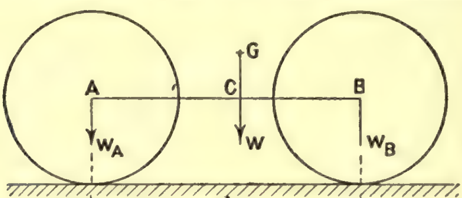


FIG. 154.

Suppose that the wheels are the same diameter, and that the centre of each wheel-axle, A and B (Fig. 154), stands

at the same height above a level floor, the wheels being locked in the same vertical plane.

When standing level, let W_A = weight exerted by the front wheel on a weighing machine table; let W_B = weight exerted by the back wheel on a weighing machine table; then—

$$W_A + W_B = \text{weight of bicycle}$$

Let AB, the horizontal distance apart of the axle centres, be d inches. If the vertical line through the c.g. G cuts AB in C, then—

$$BC = \frac{W_A}{W_A + W_B} \cdot d (\text{Art. } 87)$$

Next, let the weight exerted by the front wheel, when A stands a distance "h" inches (vertically) above B, be W_a; and let CG, the distance of the c.g. of the bicycle above AB, be H.

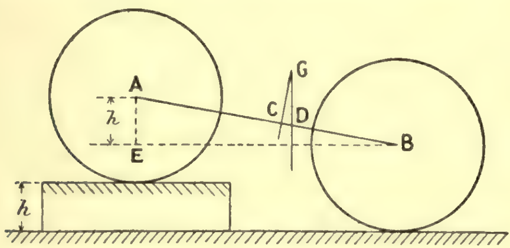


FIG. 155.

Then, since ABE and DGC (Fig. 155) are similar triangles—

$$\frac{GC}{CD} = \frac{BE}{AE} = \frac{\sqrt{(d^2 - h^2)}}{h}$$

$$\text{and } CD = BC - BD = \frac{W_A}{W_A + W_B} \cdot d - \frac{W_a}{W_A + W_B} \cdot d = \frac{W_A - W_a}{W_A + W_B} \cdot d$$

$$\text{hence GC or H} = \frac{\sqrt{(d^2 - h^2)}}{h} \cdot \frac{W_A - W_a}{W_A + W_B} \cdot d$$

In an experiment on a certain bicycle the quantities were $d = 44$ inches, $h = 6$ inches, weight of bicycle = 32·90 lbs., pressure (W_A) exerted by the front wheel when the back wheel

was on the same level = 14·50 lbs., pressure (W_a) exerted by the front wheel when the back wheel was 6 inches lower = 13·84 lbs.

$$\text{Hence } H = \frac{\sqrt{(44^2 - 6^2)}}{6} \times \frac{14·50 - 13·84}{32·90} \times 44$$
$$= 6·54 \text{ inches}$$

or the height of the c.g. above the ground is 6·54 inches plus the radius of the wheels. The distance BC of the c.g. horizontally in front of the back axle is $\frac{14·50}{32·90} \times 44$, or 19·4 inches. A similar method may be applied to motor cars or locomotives. In the latter case, all the wheels on one side rest on a raised rail on a weighing machine, thus tilting the locomotive sideways.

EXAMPLES XVI.

1. A beam rests on two supports at the same level and 12 feet apart. It carries a distributed load which has an intensity of 4 tons per foot-run at the right-hand support, and decreases uniformly to zero at the left-hand support. Find the pressures on the supports at the ends.

2. The span of a simply supported horizontal beam is 24 feet, and along three-quarters of this distance there is a uniformly spread load of 2 tons per foot run, which extends to one end of the beam: the weight of the beam is 5 tons. Find the vertical supporting forces at the ends.

3. A beam is supported at the two ends 15 feet apart. Reckoning from the left-hand end, the first 4 feet carry a uniformly spread load of 1 ton per foot run; the first 3 feet starting from the right-hand end carry a load of 6 tons per foot run evenly distributed, and in the intermediate portion the intensity of loading varies uniformly from that at the right-hand end to that at the left-hand end. Find the reaction of the supports.

4. The altitude of a cone of homogeneous material is 18 inches, and the diameter of its base is 12 inches. What is the greatest inclination on which it may stand in equilibrium on its base?

5. A cylinder is to be made to contain 250 cubic inches of material. What is the greatest height it may have in order to rest with one end on a plane inclined at 15° to the horizontal, and what is then the diameter of the base?

6. A solid consists of a hemisphere and a cylinder, each 10 inches diameter, the centre of the base of the hemisphere being at one end of the axis of the cylinder. What is the greatest length of cylinder consistent with stability of equilibrium when the solid is resting with its curved end on a horizontal plane?

7. A solid is made up of a hemisphere of iron of 3 inches radius, and a cylinder of aluminium 6 inches diameter, one end of which coincides with the plane circular face of the hemisphere. The density of iron being three times that of aluminium, what must be the length of the cylinder if the solid is to rest on a horizontal plane with any point of the hemispherical surface in contact?

8. A uniform chain, 40 feet long and weighing 10 lbs. per foot, hangs vertically. How much work is necessary to wind it up?

9. A chain weighing 12 lbs. per foot and 70 feet long hangs over a (frictionless) pulley with one end 20 feet above the other. How much work is necessary to bring the lower end to within 2 feet of the level of the higher one?

10. A chain hanging vertically consists of two parts: the upper portion is 100 feet long and weighs 16 lbs. per foot, the lower portion is 80 feet long and weighs 12 lbs. per foot. Find the work done in winding up (*a*) the first 70 feet of the chain, (*b*) the remainder.

11. A hollow cylindrical boiler shell, 7 feet internal diameter and 25 feet long, is fixed with its axis horizontal. It has to be half filled with water from a reservoir, the level of which remains constantly 4 feet below the axis of the boiler. Find how much work is required to lift the water, its weight being 62·5 lbs. per cubic foot.

12. A cubical block of stone of 3-feet edge rests with one face on the ground: the material weighs 150 lbs. per cubic foot. How much work is required to tilt the block into a position of unstable equilibrium resting on one edge?

13. A cone of altitude 2 feet rotates about a diameter of its base at a uniform speed of 180 revolutions per minute. If the weight of the cone is 20 lbs., what centrifugal pull does it exert on the axis about which it rotates?

14. A shaft making 150 rotations per minute has attached to it a pulley weighing 80 lbs., the c.g. of which is 0·1 inch from the axis of the shaft. Find the outward pull which the pulley exerts on the shaft.

15. The arc of a circle of 8 inches radius subtends an angle of 60° at the centre. Find the area of the surface generated when this arc revolves about its chord; find also the volume of the solid generated by the revolution of the segment about the chord.

16. A groove of V-shaped section, 1·5 inches wide and 1 inch deep, is cut in a cylinder 4 inches in diameter. Find the volume of the material removed.

17. A symmetrical rectangular table, the top of which measures 8 feet by 3 feet, weighs 150 lbs., and is supported by castors at the foot of each leg, each castor resting in contact with a level floor exactly under a corner of the table top. Two of the legs 3 feet apart are raised 10 inches on to the plate of a weighing machine, and the pressure exerted by them is 66·5 lbs. Find the height of the c.g. of the table above the floor when the table stands level.

CHAPTER IX

MOMENTS OF INERTIA—ROTATION

133. Moments of Inertia.

(1) *Of a Particle.*—If a particle P (Fig. 156), of weight w and mass $\frac{w}{g}$, is situated at a distance r from an axis OO′, then its moment of inertia about that axis is defined as the quantity $\frac{w}{g} \cdot r^2$, or (mass of P) × (distance from OO′)^2.

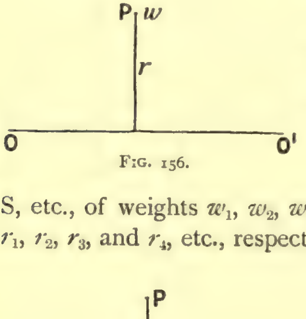
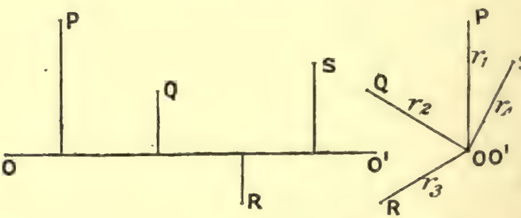


FIG. 156.

(2) *Of Several Particles.*—If several particles, P, Q, R, and S, etc., of weights w_1, w_2, w_3, w_4, etc., be situated at distances r_1, r_2, r_3, and r_4, etc., respectively from an axis OO′ (Fig. 157),



End view of axis OO′.

FIG. 157.

then the total moment of inertia of the several particles about that axis is defined as—

$$\frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 + \frac{w_4}{g}r_4^2 +, \text{etc.}$$

$$\text{or } \Sigma\left(\frac{w}{g}r^2\right)$$

or $\Sigma\{$(mass of each particle) $\times$ (its distance from $OO')^2\}$

(3) *Rigid Bodies.*—If we regard a rigid body as divisible into a very large number of parts, each so small as to be regarded as a particle, then the moment of inertia of the rigid body about any axis is equal to the moment of inertia of such a system of particles about that axis. Otherwise, suppose a body is divided into a large but finite number of parts, and the mass of each is multiplied by the square of the distance of some point in it from a line OO′; the sum of these products will be an approximation to the moment of inertia of the whole body. The approximation will be closer the larger the number of parts into which the body is divided; as the number of parts is indefinitely increased, and the mass of each correspondingly decreased, the sum of the products tends towards a fixed limiting value, which it does not exceed however far the subdivision be carried. This limiting sum is the moment of inertia of the body, which may be written $\Sigma(mr^2)$ or $\Sigma\left(\frac{w}{g} \cdot r^2\right)$.

Units.—The units in which a moment of inertia is stated depend upon the units of mass and length adopted. No special names are given to such units. The "engineer's unit" or gravitational unit is the moment of inertia about an axis of unit mass (32·2 lbs.) at a distance of 1 foot from the axis.

134. Radius of Gyration.—The radius of gyration of a body about a given axis is that radius at which, if an equal mass were concentrated, it would have the same moment of inertia.

Let the moment of inertia $\Sigma\left(\frac{w}{g}r^2\right)$ of a body about some axis be denoted by I, and let its total weight $\Sigma(w)$ be W, and therefore its total mass $\Sigma\left(\frac{w}{g}\right) = \frac{W}{g}$.

Let k be its radius of gyration about the same axis. Then, from the above definition—

$$I = \frac{W}{g}k^2 = \Sigma\left(\frac{w}{g}r^2\right)$$
$$\text{and } k^2 = \frac{Ig}{W} = \frac{\Sigma(wr^2)}{W}$$

135. Moments of Inertia of a Lamina about an Axis perpendicular to its Plane.

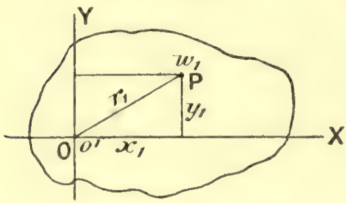


FIG. 158.

Let the distances of any particle, P (Fig. 158), of a lamina from two perpendicular axes, OY and OX, in its plane be x_1 and y_1 respectively, and let w_1 be its weight, and r_1 its distance from O, so that $r_1^2 = x_1^2 + y_1^2$.

Then, if I_X and I_Y denote the moments of inertia of the lamina made up of such particles, about OX and OY respectively—

$$I_X = \frac{w_1}{g}y_1^2 + \frac{w_2}{g}y_2^2 + \frac{w_3}{g}y_3^2 +, \text{etc.}$$

$$I_Y = \frac{w_1}{g}x_1^2 + \frac{w_2}{g}x_2^2 + \frac{w_3}{g}x_3^2 +, \text{etc.}$$

and adding—

$$I_X + I_Y = \left\{\frac{w_1}{g}(x_1^2 + y_1^2) + \frac{w_2}{g}(x_2^2 + y_2^2) + \frac{w_3}{g}(x_3^2 + y_3^2) +, \text{etc.}\right\}$$
$$= \frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 +, \text{etc.}$$

or $\Sigma\left(\frac{w}{g}r^2\right)$, which may be denoted by I_0.

$$\text{Then } I_0 = I_X + I_Y \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This quantity I_0 is by definition the moment of inertia about an axis OO′ perpendicular to the plane of the lamina, and through O the point of intersection of OX and OY.

Hence the sum of the moments of inertia of a lamina about any two mutually perpendicular axes in its plane, is equal to the moment of inertia about an axis through the intersection of the other two axes and perpendicular to the plane of the lamina.

Also, if k_X, k_Y, and k_0 be the radii of gyration about OX, OY, and OO′ respectively, OO′ being perpendicular to the plane of Fig. 158, and if $\Sigma\left(\frac{w}{g}\right) = \frac{W}{g}$, the mass of the whole lamina—

$$\Sigma\left(\frac{w}{g}r^2\right) \text{ or } I_0 = k_0^2 \cdot \frac{W}{g}$$

$$\text{and } I_X = k_X^2 \cdot \frac{W}{g}$$

$$\text{and } I_Y = k_Y^2 \cdot \frac{W}{g}$$

and therefore, since $I_X + I_Y = k_0^2 \cdot \frac{W}{g}$ by (1)

$$k_0^2 = k_X^2 + k_Y^2 \quad . \quad . \quad . \quad . \quad (2)$$

Or, in words, the sum of the squares of the radii of gyration of a lamina about two mutually perpendicular axes in its plane, is equal to the square of its radius of gyration about an axis through the intersection of the other two axes and perpendicular to the plane of the lamina.

136. Moments of Inertia of a Lamina about Parallel Axes in its Plane.—Let P, Fig. 159, be a constituent particle of weight w_1 of a lamina, distant x_1 from an axis *ZZ′* in the plane of the lamina and through G, the c.g. of the lamina, the distances being reckoned positive to the right and negative to the left of *ZZ′*. Let OO′ be an axis in the plane of the lamina parallel to *ZZ′* and distant d from it. Then the distance of P from OO′ is $d - x_1$, whether P is to the right or left of *ZZ′*.

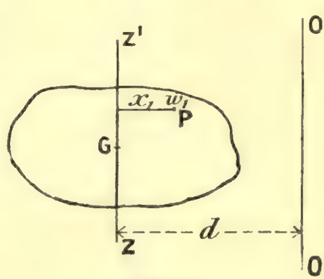


FIG. 159.

Let I_0 be the moment of inertia of the lamina about OO′;
and let I_Z " " " " ZZ′.
Then—

$$I_0 = \left\{\frac{w_1}{g}(d - x_1)^2 + \frac{w_2}{g}(d - x_2)^2 + \frac{w_3}{g}(d - x_3)^2 +, \text{etc.}\right\}$$

$$I_0 = d^2\left(\frac{w_1}{g}+\frac{w_2}{g}+\frac{w_3}{g}+,\text{etc.}\right) + \left(\frac{w_1}{g}x_1^2 + \frac{w_2}{g}x_2^2 + \frac{w_3}{g}x_3^2 +,\text{etc.}\right)$$
$$- 2\frac{d}{g}(w_1x_1 + w_2x_2 + w_3x_3 +, \text{etc.})$$

The sum $w_1x_1 + w_2x_2 + w_3x_3 +$, etc., is, by Art 114, equal to

$(w_1 + w_2 + w_3 +$, etc.$) \times$ (distance of c.g. from ZZ′)

which is zero, since the second factor is zero. Hence—

$$I_0 = \frac{d^2}{g}(w_1+w_2+w_3+,\text{etc.}) + \left(\frac{w_1}{g}x_1^2+\frac{w_2}{g}x_2^2+\frac{w_3}{g}x_3^2+,\text{etc.}\right)$$
$$= d^2\Sigma\left(\frac{w}{g}\right) + I_Z$$

$$\text{or } I_0 = \frac{W}{g}d^2 + I_Z \quad \ldots \ldots \quad (1)$$

where W is the total weight of the lamina. And dividing each term of this equation by $\frac{W}{g}$—

$$k_0^2 = d^2 + k_Z^2 \quad \ldots \ldots \quad (2)$$

where k_0 and k_Z are the radii of gyration about OO′ and ZZ′ respectively.

137.[1] **Extension of the Two Previous Articles to Solid Bodies.**—(*a*) Let ZX and ZY (Fig. 160) represent (by their traces) two planes perpendicular to the plane of the paper and to each other, both passing through the c.g. of a solid body.

Let P be a typical particle of the body, its weight being w_1,

[1] This article may be omitted on first reading. The student acquainted with the integral calculus will readily apply the second theorem to simple solids.

and its distances from the planes ZY and ZX being x_1 and y_1 respectively. Then, if r_1 is the distance of P from an axis ZZ′, which is the intersection of the planes XZ and YZ, and passes through the c.g. $r_1^2 = x_1^2 + y_1^2$.

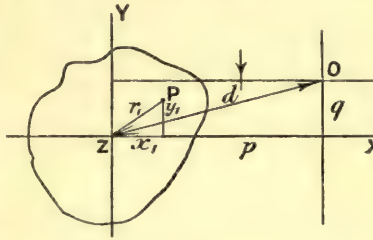


Fig. 160.

Let I_Z be the moment of inertia of the body about ZZ′, and I_0 that about a parallel axis OO′. Let OO′ be distant d from ZZ′, and distant p and q from planes ZY and ZX respectively. Then $p^2+q^2=d^2$.

Let other constituent particles of the body of weights w_2, w_3, w_4, etc., be at distances x_2, x_3, x_4, etc., from the plane ZY, and distances y_2, y_3, y_4, etc., from the plane ZX respectively, the x distances being reckoned positive to the right and negative to the left of ZY, and the y distances being reckoned positive above and negative below ZX. Let r_2, r_3, r_4, etc., be the distances of the particles from ZZ_1. Let $w_1 + w_2 + w_3 +$, etc. $= \Sigma(w)$ or W, the total weight of the body.

By definition—

$$I_0 = \frac{1}{g}\Sigma(w_1 . OP^2)$$

$$\text{and } OP_1^2 = (p - x_1)^2 + (q - y_1)^2$$

$$\text{therefore } I_0 = \frac{1}{g}\{w_1(p - x_1)^2 + w_1(q - y_1)^2 + w_2(p - x_2)^2 + w_2(q-y_2)^2 + w_3(p-x_1)^2 + w_3(q - y_3)^2 +, \text{etc.}\}$$

$$I_0 = \frac{1}{g}\{p^2(w_1+w_2+w_3+, \text{etc.}) + q^2(w_1+w_2+w_3+, \text{etc.}) + w_1(x_1^2+y_1^2) + w_2(x_2^2+y_2^2) + w_3(x_3^2+y_3^2)+, \text{etc.} - 2p(w_1x_1+w_2x_2+w_3x_3+, \text{ etc.}) - 2q(w_1y_1+w_2y_2 + w_3y_3 +, \text{etc.})\}$$

$$I_0 = \frac{1}{g}p^2W + q^2W + (w_1r_1^2 + w_2r_2^2 + w_3r_3^2 +, \text{ etc.}) - 2p\Sigma(wx) - 2q\Sigma(wy)\}$$

$$= \frac{1}{g}\{(p^2 + q^2)W + \Sigma(wr^2) - 2p\Sigma(wx) - 2q\Sigma(wy)$$

$$\text{and } p^2 + q^2 = d^2$$

$$\frac{1}{g}\Sigma(wr^2) = I_Z$$

$$\Sigma(wx) = \Sigma(wy) = 0$$

since the planes XZ and YZ pass through the c.g. of the body (Art. 113).

$$\text{Hence } I_0 = \frac{W}{g}d^2 + I_Z \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and dividing both sides of (1) by $\frac{W}{g}$—

$$k_0^2 = d^2 + k_Z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where k_0 = radius of gyration about OO′, and k_Z = radius of gyration about ZZ′.

(*b*) Also—

$$I_Z = \frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 +, \text{etc.}$$

$$= \frac{w_1}{g}(x_1^2 + y_1^2) + \frac{w_2}{g}(x_2^2 + y_2^2) + \frac{w_3}{g}(x_3^2 + y_1^2) +, \text{etc.}$$

$$= \frac{1}{g}(w_1x_1^2 + w_2x_2^2 + w_3x_3^2 +, \text{etc.}) + \frac{1}{g}(w_1y_1^2 + w_2y_2^2 + w_3y_3^2 +, \text{etc.})$$

$$\frac{W}{g}k_Z^2 = \frac{1}{g}\Sigma(wx^2) + \frac{1}{g}\Sigma(wy^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$k_Z^2 = \frac{\Sigma(wx^2)}{W} + \frac{\Sigma(wy^2)}{W}$$

which may be written—

$$k_Z^2 = \overline{x^2} + \overline{y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\overline{x^2}$ and $\overline{y^2}$ are the *mean squares* of the distances of the body from the planes YZ and XZ respectively. The two quantities $\overline{x^2}$ and $\overline{y^2}$ are in many solids easily calculated.

138. Moment of Inertia of an Area.—The moment of inertia I_0 of a lamina about a given axis OO′ in its plane is $\Sigma\left(\frac{w}{g}r^2\right)$ (Art. 133), where w is the weight of a constituent

particle, and r its distance from the axis OO′. This quantity is equal to $\frac{W}{g} \cdot k^2$ (Art. 134), where k is the radius of gyration about this axis OO′, and W is the total weight of the lamina, so that—

$$k^2 = \frac{\Sigma\left(\frac{w}{g} r^2\right)}{\frac{W}{g}} \text{ or } \frac{\Sigma(wr^2)}{W}$$

In a thin lamina of uniform thickness t, the area a (Fig. 161) occupied by a particle of weight w is proportional to w, for $w = a \cdot t \cdot D$, where D is the weight per unit volume of the material;

hence $\Sigma(wr^2) = tD\Sigma(ar^2)$

and similarly, $W = A \cdot t \cdot D$, where A is the total area of the lamina;

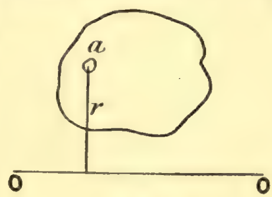


FIG. 161.

$$\text{hence } k^2 = \frac{tD\Sigma(a \cdot r^2)}{At \cdot D} = \frac{\Sigma(a \cdot r^2)}{A}$$

Thus the thickness and density of a lamina need not be known in order to find its radius of gyration, and an *area* may properly be said to have a radius of gyration about a given axis.

The quantity $\Sigma(ar^2)$ is also spoken of as the *moment of inertia of the area* of the lamina about the axis OO′ from which a portion a is distant r.

The double use of this term "moment of inertia" is unfortunate. The "moment of inertia of an area" $\Sigma(ar^2)$ or $k^2 \cdot A$ is not a true moment of inertia in the sense commonly used in mechanics, viz. that of Art. 133; it must be multiplied by the factor "mass per unit area" to make it a true moment of inertia. As before mentioned, the area has, however, a radius of gyration about an axis OO′ in its plane defined by the equation—

$$k^2 = \frac{\Sigma(ar^2)}{A}$$

Units.—The units of the geometrical quantity $\Sigma(ar^2)$, called *moment of inertia of an area*, depend only upon the units of length employed. If the units of length are inches, a moment of inertia of an area is written (inches)^4.

139. Moment of Inertia of Rectangular Area about Various Axes.—Let ABCD (Fig. 162) be a rectangle, AB $= d$, BC $= b$. The moment of inertia of the area ABCD about the axis OO′ in the side AD may be found as follows. Suppose AB divided into a large number n, of equal parts, and the area ABCD divided into n equal narrow strips, each of width $\frac{d}{n}$. The whole of any one strip EFGH is practically at a distance, say, FA from AD, and if EFGH is the pth strip from AD, FA $= p \times \frac{d}{n}$.

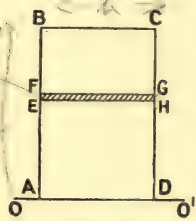


Fig. 162.

Multiplying the area EFGH, viz. $b \times \frac{d}{n}$, by the square of its distance from AD, we have—

$$(\text{area EFGH}) \times \text{FA}^2 = b \times \frac{d}{n} \times \left(\frac{pd}{n}\right)^2 = bd\frac{p^2d^2}{n^3} = \frac{bp^2d^3}{n^3}$$

There are n such strips, and therefore the sum of the products of the areas multiplied by the squares of their distances from OO′, which may be denoted by $\Sigma(ar^2)$, is—

$$\frac{bd^3}{n^3}(1^2 + 2^2 + 3^2 + 4^2 + \ . \ . \ . \ + p^2 + \ . \ . \ . \ + n^2)$$

$$\text{or } \Sigma(ar^2) = \frac{bd^3}{n^3} \times \frac{n(n+1)(2n+1)}{6} = \frac{bd^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

When n is indefinitely great, $\frac{3}{n} = 0$, and $\frac{1}{n^2} = 0$, and the sum $\Sigma(ar^2)$ becomes $\frac{bd^3}{6} \times 2$ or $\frac{bd^3}{3}$. This is the "moment of inertia of the area" about OO′; or, the radius of gyration of the area about OO′ being k—

$$k^2 = \frac{\Sigma(ar^2)}{\Sigma(a)} = \tfrac{1}{3}bd^3 \div bd = \tfrac{1}{3}d^2$$

If ABCD is a lamina of uniform thickness of weight w, its true moment of inertia about OO′ is $\frac{w}{g}k^2 = \frac{1}{3}\frac{w}{g}.d^2$.

The radius of gyration of the same area ABCD about an axis PQ (Fig. 163) in the plane of the figure and parallel to OO′ and distant $\frac{d}{2}$ from it, dividing the rectangle into halves, can be found from the formula (2), Art. 136, viz.—

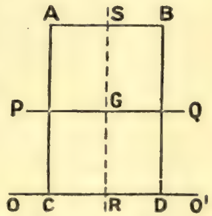


FIG. 163.

$$k_0^2 \text{ or } \tfrac{1}{3}d^2 = k_P^2 + \left(\frac{d}{2}\right)^2$$

where k_P = radius of gyration about PQ;

$$\text{whence } k_P^2 = (\tfrac{1}{3} - \tfrac{1}{4})d^2 = \tfrac{1}{12}d^2$$

The sum $\Sigma(ar^2)$ about PQ is then $\Sigma(a) \times k_P^2 = bd \times \frac{d^2}{12} = \frac{1}{12}bd^3$

Similarly, if k_S is the radius of gyration of the rectangle about RS—

$$k_S^2 = \tfrac{1}{12}b^2$$

and therefore, if k_G = radius of gyration about an axis through G (the c.g.) and perpendicular to the figure—

$$k_G^2 = k_S^2 + k_P^2 \text{ or } \tfrac{1}{12}(b^2 + d^2) \text{ (Art. 135 (2))}$$

which is also equal to $\frac{1}{12}\text{BC}^2$ or $\frac{1}{3}\text{GB}^2$.

Example.—A plane figure consists of a rectangle 8 inches by 4 inches, with a rectangular hole 6 inches by 3 inches, cut so that the diagonals of the two rectangles are in the same straight lines. Find the geometrical moment of inertia of this figure, and its radius of gyration, about one of the short outer sides.

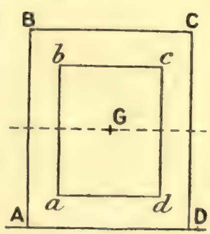


FIG. 164.

Let I_A be the moment of inertia of the figure about AD (Fig. 164), and k be its radius of gyration about AD.

Moment of inertia of *abcd* about AD $= \frac{1}{12}(\text{area } abcd) \times (\text{side } ab)^2 + (\text{area } abcd) \times (\tfrac{1}{2}\text{AB})^2$ (Arts. 139 and 136)

Moment of inertia of ABCD about AD $= \frac{1}{3} \times 4 \times 8^3$

Hence I_A = moment of inertia of ABCD − moment of inertia of *abcd*

$$= \frac{1}{3} \cdot 4 \cdot 8^3 - \left(\frac{1}{12} \times 6^3 \times 3 + 6 \times 3 \times 4^2\right)$$
$$= \frac{2048}{3} - (54 + 288) = 340\cdot6 \text{ (inches)}^4$$

The area of the figure is—

$$8 \times 4 - 6 \times 3 = 14 \text{ square inches}$$
$$\text{therefore } k^2 = \frac{340\cdot6}{14} = 24\cdot33 \text{ (inches)}^2$$
$$\text{and } k = 4\cdot93 \text{ inches}$$

140. Moment of Inertia of a Circular Area about Various Axes.—(1) About an axis OO′ through O, its centre, and perpendicular to its plane.

Let the radius OS of the circle (Fig. 165) be equal to R.

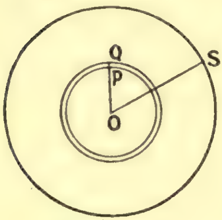


FIG. 165.

Suppose the area divided into a large number n, of circular or ring-shaped strips such as PQ, each of width $\frac{R}{n}$. Then the distance of the pth strip from O is approximately $p \times \frac{R}{n}$, and its area is approximately—

$$2\pi \times \text{radius} \times \text{width} = 2\pi \times p\frac{R}{n} \cdot \frac{R}{n} = 2\pi p\frac{R^2}{n^2}$$

The moment of inertia of this strip of area about OO′ is then—

$$2\pi p\frac{R^2}{n^2} \times \left(\frac{pR}{n}\right)^2 = 2\pi\frac{R^4}{n^4}p^3$$

and adding the sum of all such quantities for all the n strips—

$$\Sigma(ar^2) = 2\pi\frac{R^4}{n^4}(1^3 + 2^3 + 3^3 + 4^3 + \ldots p^3 + \ldots + n^3)$$
$$= 2\pi\frac{R^4}{n^4}\left\{\frac{n(n+1)}{2}\right\}^2 = 2\pi \cdot \frac{R^4}{n^4} \cdot \frac{n^4 + 2n^3 + n^2}{4}$$
$$= \frac{\pi \cdot R^4}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

When n is indefinitely great, $\frac{2}{n} = 0$ and $\frac{1}{n^2} = 0$, and the

sum $\Sigma(ar^2)$ becomes $\frac{\pi R^4}{2}$, which is the "moment of inertia of the circular area" about OO′.

And since $\Sigma(ar^2)$ about $OO' = \frac{\pi R^4}{2}$, if we divide each side of the equation by the area (πR^2) of the circle—

$$k_0{}^2 \pi R^2 = \frac{\pi R^4}{2}$$

$$k_0{}^2 = \frac{R^2}{2}$$

where k_0 is the radius of gyration of the circular area about an axis OO′ through its centre and perpendicular to its plane.

(2) About a diameter.

Again, if k_A and k_C are the radii of gyration of the same area about the axes AB and CD respectively (Fig. 166)—

$$k_A{}^2 + k_C{}^2 = k_0{}^2 = \frac{R^2}{2} \text{ (Art. 135 (2))}$$

$$\text{hence } k_A{}^2 = k_C{}^2 = \tfrac{1}{2} \cdot \frac{R^2}{2} = \frac{R^2}{4}$$

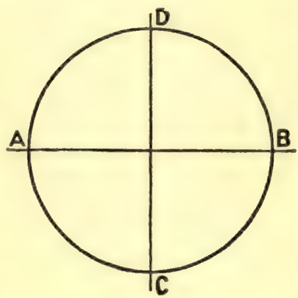


FIG. 166.

from which the relations between the moments of inertia about AB, DC, and OO′ may be found by multiplying each term by πR^2. That is, the moment of inertia of the circular area about a diameter is half that about an axis through O and perpendicular to its plane.

Example.—Find the radius of gyration of a ring-shaped area, bounded outside by a circle of radius a, and inside by a concentric circle of radius b, about a diameter of the outer circle.

The moment of inertia of the area bounded by the outer circle, about AB (Fig. 167) is $\frac{\pi a^4}{4}$; that of the inner circular area about

the same line is $\frac{\pi b^4}{4}$; hence that of the ring-shaped area is $\frac{\pi}{4}(a^4 - b^4)$. The area is $\pi(a^2 - b^2)$; hence, if k is the radius of gyration of the ring-shaped area about AB—

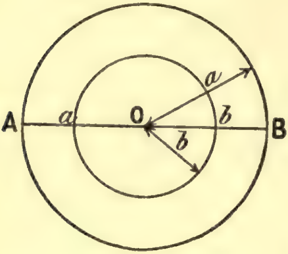


Fig. 167.

$$k^2 = \frac{\pi}{4}(a^4 - b^4) \div \pi(a^2 - b^2)$$
$$= \frac{a^2 + b^2}{4}$$

Note that $k_0^2 = \frac{a^2 + b^2}{2} = \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$, so that when a and b are nearly equal, *i.e.* when $a - b$ is a small quantity, the radius of gyration k_0, about the axis O, approaches the arithmetic mean $\frac{a+b}{2}$ of the inner and outer radii.

141. Moment of Inertia of a Thin Uniform Rod.—The radius of gyration of a thin rod d units long and of uniform material, about an axis through one end and perpendicular to the length of the rod, will evidently be the same as that of a narrow rectangle d units long, which, by Art. 139, is given by the relation $k^2 = \frac{1}{3}d^2$, where k is the required radius of gyration. Hence, if the weight of the rod is W lbs., its moment of inertia about one end is $\frac{W}{g}k^2$ or $\frac{W}{g} \cdot \frac{d^2}{3}$.

Similarly, its moment of inertia about an axis through the middle point and perpendicular to the length is $\frac{W}{g} \cdot \frac{d^2}{12}$.

142. Moment of Inertia of a Thin Circular Hoop.—(1) The radius of the hoop being R, all the matter in it is at a distance R from the centre of the hoop. Hence the radius of gyration about an axis through O, the centre of the hoop, and perpendicular to its plane, is R, and the moment of inertia about this axis is $\frac{w}{g}$. R^2, where W is the weight of the hoop.

(2) The radius of gyration about diameters OX and OY (Fig. 168) being k_x and k_y respectively—

$$R^2 = k_x^2 + k_y^2 \text{ (Art. 135 (2))}$$

hence $k_x^2 = k_y^2 = \dfrac{R^2}{2}$

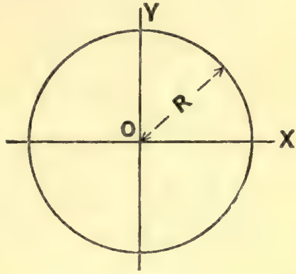


FIG. 168.

and the moment of inertia about any diameter of the hoop is $\dfrac{W}{g} \cdot \dfrac{R^2}{2}$.

143. Moment of Inertia of Uniform Solid Cylinder.—(1) About the axis OO′ of the cylinder. The cylinder may be looked upon as divided into a large number of circular discs (Fig. 169) by planes perpendicular to the axis of the cylinder. The radius of gyration of each disc about the axis of the cylinder is given by the relation $k^2 = \dfrac{R^2}{2}$, where k is radius of gyration of the disc, and R the outside radius of the cylinder and discs. If the weight of any one disc is w, and that of the whole cylinder is W, the moment of inertia of one disc is—

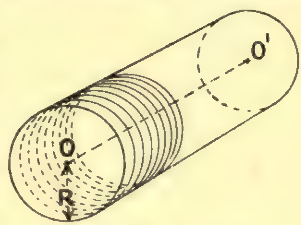


FIG. 169.

$$\frac{w}{g} \cdot \frac{R^2}{2}$$

and that of the whole cylinder is—

$$\Sigma\left(\frac{w}{g} \cdot \frac{R^2}{2}\right) = \frac{R^2}{2g}\Sigma(w) = \frac{W}{g} \cdot \frac{R^2}{2}$$

and the square of the radius of gyration of the cylinder is $\dfrac{R^2}{2}$.

(2) **About an Axis perpendicular to that of the Cylinder and through the Centre of One End.**—Let OX (Fig. 170) be the axis about which the moment of inertia

of the cylinder is required. Let R be the radius of the cylinder, and l its length.

Let $\overline{x^2}$ = the mean square of the distance of the constituent particles from the plane YOO′Y′;

$\overline{y^2}$ = the mean square of the distance of the constituent particles from the plane OXX′O′;

k_0 = the radius of gyration of the cylinder about OO′.

$$\text{Then } k_0^2 = \overline{x^2} + \overline{y^2} \text{ by Art. 137 (4)}$$

and from the symmetry of the solid, $\overline{x^2} = \overline{y^2}$;

$$\text{hence } k_0^2 \text{ or } \frac{R^2}{2} = 2\overline{x^2} = 2\overline{y^2}$$

$$\text{and } \overline{x^2} = \frac{R^2}{4} = \overline{y^2}$$

The cylinder being supposed divided into thin parallel rods all parallel to the axis and l units long, the mean square of the

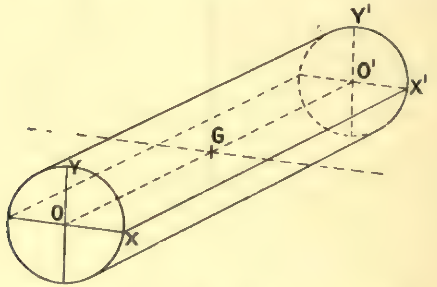


FIG. 170.

distance of the particles forming the rod from the plane YOX of one end, is the same as the square of the radius of gyration of a rod of length l about an axis perpendicular to its length and through one end, viz. $\frac{l^2}{3}$ (Art. 141). The axis OX is the intersection of the planes XOO′X′ and YOX, the end plane; hence, if k_X is the radius of gyration about OX—

$$k_X^2 = \overline{y^2} + \frac{l^2}{3} = \frac{R^2}{4} + \frac{l^2}{3} \text{ (Art. 137 (2))}$$

(3) Also, if k_G is the radius of gyration about a parallel axis through G, the c.g. of the cylinder—

$$k_X^2 = k_G^2 + \left(\frac{l}{2}\right)^2 \text{ (Art. 137 (2))}$$

$$\text{or } k_G^2 = k_X - \frac{l^2}{4} = \frac{R^2}{4} + \frac{l^2}{12}$$

The moments of inertia of the cylinder about these various axes are to be found by multiplying the square of the radius of gyration about that axis by the mass $\frac{w}{g}$, where w is the weight of the cylinder, in accordance with the general relation I $= \frac{w}{g}k^2$ (Art. 134).

Example.—A solid disc flywheel of cast iron is 10 inches in diameter and 2 inches thick. If the weight of cast iron is 0·26 lb. per cubic inch, find the moment of inertia of the wheel about its axis in engineers' units.

The volume of the flywheel is $\pi \times 5^2 \times 2 = 50\pi$ cubic inches
the weight is then $0·26 \times 50\pi = 40·9$ lbs.
and the mass is $\frac{40·9}{32·2} = 1·27$ units

The square of the radius of gyration is $\frac{1}{2}(\frac{5}{12})^2$ (feet)^2. Therefore the moment of inertia is—

$$1·27 \times \frac{25}{288} = 0·1104 \text{ unit}$$

Examples XVII.

1. A girder of **I**-shaped cross-section has two horizontal flanges 5 inches broad and 1 inch thick, connected by a vertical web 9 inches high and 1 inch thick. Find the "moment of inertia of the area" of the section about a horizontal axis through its c.g.

2. Fig. 171 represents the cross-section of a cast-iron girder. AB is 4 inches, BC 1 inch, EF 1 inch; EH is 6 inches, KL is 8 inches, and KN is 1·5 inches. Find the moment of inertia and radius of gyration of the area of the section about the line NM.

3. Find, from the results of Ex. 2, the moment of inertia and radius of gyration of the area of section about an axis through the c.g. of the section and parallel to NM.

4. Find the moment of inertia of the area enclosed between two concentric circles of 10 inches and 8 inches diameter respectively, about a diameter of the circles.

5. Find the radius of gyration of the area bounded on the outside by a circle 12 inches diameter, and on the inside by a concentric circle of 10 inches diameter, about an axis through the centre of the figure and perpendicular to its plane.

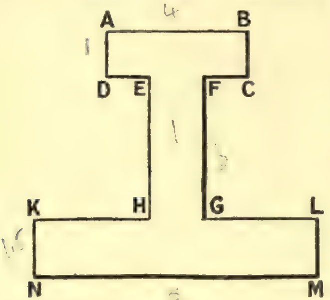


Fig. 171.

6. The pendulum of a clock consists of a straight uniform rod, 3 feet long and weighing 2 lbs., attached to which is a disc 0·5 foot in diameter and weighing 4 lbs., so that the centre of the disc is at the end of the rod. Find the moment of inertia of the pendulum about an axis perpendicular to the rod and to the central plane of the disc, passing through the rod 2·5 feet from the centre of the disc.

7. Find the radius of gyration of a hollow cylinder of outer radius a and inner radius b about the axis of the cylinder.

8. Find the radius of gyration of a flywheel rim 3 feet in external diameter and 4 inches thick, about its axis. If the rim is 6 inches broad, and of cast-iron, what is its moment of inertia about its axis? Cast iron weighs 0·26 lb. per cubic inch.

144. Kinetic Energy of Rotation.—If a particle of a body weighs w_1 lbs., and is rotating with angular velocity ω about a fixed axis r_1 feet from it, its speed is ωr_1 feet per second (Art. 33), and its kinetic energy is therefore $\frac{1}{2} \cdot \frac{w_1}{g} \cdot (\omega r_1)^2$ foot-lbs. (Art. 60). Similarly, another particle of the same rigid body situated r_2 feet from the fixed axis of rotation, and weighing w_2 lbs., will have kinetic energy equal to $\frac{1}{2} \cdot \frac{w_2}{g} \cdot (\omega r_2)^2$; and if the whole body is made up of particles weighing w_1, w_2, w_3, w_4, etc., lbs., situated at r_1, r_2, r_3, r_4, etc., feet respectively from the axis of rotation, the total kinetic energy of the body will be—

$$\frac{1}{2}\left\{\frac{w_1}{g}(\omega r_1)^2 + \frac{w_2}{g}(\omega r_2)^2 + \frac{w_3}{g}(\omega r_3)^2 +, \text{ etc.}\right\}$$

$$\text{or } \frac{1}{2}\omega^2\left(\frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 +, \text{ etc.}\right) \text{ foot-lbs.}$$

The quantity $\left(\frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 +, \text{etc.}\right)$ or $\Sigma\left(\frac{w}{g}r^2\right)$ has been defined (Art. 133) as the moment of inertia I, of the body about the axis. Hence the kinetic energy of the body is $\frac{1}{2}I\omega^2$, or $\frac{1}{2}\cdot\frac{W}{g}K^2\omega^2$, or $\frac{1}{2}\frac{W}{g}V^2$ foot-lbs., where K = radius of gyration of the body in feet about the axis of rotation, and V = velocity of the body in feet per second at that radius of gyration. This is the same as the kinetic energy $\frac{1}{2}MV^2$ or $\frac{W}{2g}V^2$ of a mass M or $\frac{W}{g}$, all moving with a linear velocity V.

The kinetic energy of a body moving at a given *linear* velocity is proportional to its mass; that of a body moving about a fixed axis with given *angular* velocity is proportional to its moment of inertia. We look upon the moment of inertia of a body as its rotational inertia, *i.e.* the measure of its inertia with respect to angular motion (see Art. 36).

145. Changes in Energy and Speed.—If a body of moment of inertia I, is rotating about its axis with an angular velocity ω_1, and has a net amount of work E done upon it, thereby raising its velocity to ω_2; then, by the Principle of Work (Art. 61)—

$$\frac{1}{2}I(\omega_2^2 - \omega_1^2) = E$$

$$\text{or } \frac{1}{2}\frac{W}{g}K^2(\omega_2^2 - \omega_1^2) = E$$

$$\text{or } \frac{1}{2}\frac{W}{g}(V_2^2 - V_1^2) = E$$

where K = radius of gyration about the axis of rotation, and V_2 and V_1 are the final and initial velocities respectively at a radius K from the axis.

Hence the change of energy is equal to that of an equal weight moving with the same final and initial velocities as a point distant from the axis by the radius of gyration of the body. If the body rotating with angular velocity ω_2 about the axis is opposed by a tangential force, and does work of amount E in overcoming this force, its velocity will be reduced

to ω_1, the loss of kinetic energy being equal to the amount of work done (Art. 61).

146. Constant resisting Force.—Suppose a body, such as a wheel, has a moment of inertia I, and is rotating at an angular velocity ω_2 about an axis, and this rotation is opposed by a constant tangential force F at a radius r from the axis of rotation, which passes through the centre of gravity of the body. Then the *resultant* centripetal force on the body is zero (Art. 130). The particles of the body situated at a distance r from the centre are acted on by a resultant or effective force always in the same straight line with, and in opposite direction to, their own velocity, and therefore have a constant retardation in their instantaneous directions of motion (Art. 40). Hence the particles at a radius r have their linear velocity, and therefore also their angular velocity, decreased at a constant rate; and since, in a rigid body, the angular velocity of rotation about a fixed axis of every point is the same, the whole body suffers uniform angular retardation.

Suppose the velocity changes from ω_2 to ω_1 in t seconds, during which the body turns about the axis through an angle θ radians. The uniform angular retardation α is $\frac{\omega_2 - \omega_1}{t}$.

Also the work done on the wheel is $Fr \times \theta$ (Art. 57), hence—

$$F . r . \theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2) = \text{loss of kinetic energy} \quad . \quad (1)$$

The angle turned through during the retardation period is—

$$\theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2) \div F . r$$

Note that $F . r$ is the moment of the resisting force or the resisting torque.

$$\begin{aligned}
\text{Again, } \omega_2^2 - \omega_1^2 &= (\omega_2 + \omega_1)(\omega_2 - \omega_1) \\
\text{and } \omega_2 - \omega_1 &= \alpha t \\
\text{and } \omega_1 + \omega_2 &= \text{twice the average angular velocity} \\
&\quad\ \text{during the retardation} \\
&= \frac{2\theta}{t}
\end{aligned}$$

Hence the relation—

$$F \, . \, r \, . \, \theta = \tfrac{1}{2}I(\omega_2^2 - \omega_1^2)$$

may be written—

$$Fr\theta = \tfrac{1}{2}I \, . \, \alpha \, . \, t \, . \, \frac{2\theta}{t}$$

$$\text{or } F \, . \, r = I \, . \, \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e. the moment of the resisting force about the axis of rotation is equal to the moment of inertia of the body multiplied by its angular retardation.

Similarly, if F is a driving instead of a resisting force, the same relations would hold with regard to the rate of *increase* of angular velocity, viz. the moment of the accelerating force is equal to the moment of inertia of the body multiplied by the angular acceleration produced. Compare these results with those of Art. 40 for linear motion.

We next examine rather more generally the relation between the angular velocity, acceleration, and inertia of a rigid body.

147. Laws of Rotation of a Rigid Body about an Axis through its Centre of Gravity.—Let w be the weight of a constituent particle of the body situated at P (Fig. 172), distant r from the axis of rotation O; let ω be the angular velocity of the body about O. Then the velocity v of P is ωr.

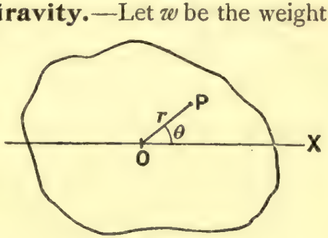


Fig. 172.

Adding the vectors representing the momenta of all such particles, we have the total momentum estimated in any particular direction, such as OX (Fig. 172), viz.—

$$\Sigma\left(\frac{w}{g}v \cos \theta\right), \text{ or } \frac{\omega}{g}\Sigma(wr \cos \theta)$$

But Σ $(wr \cos \theta)$ is zero when estimated in any direction if $r \cos \theta$ is measured from a plane through the c.g. Hence the total linear momentum resolved in any given direction is zero.

Moment of Momentum, or Angular Momentum of

a Rigid Body rotating about a Fixed Axis.—This is defined as the sum of the products of the momenta of all the particles multiplied by their respective distances from the axis, or $\Sigma\left(\frac{w}{g} \cdot v \cdot r\right)$.

$$\text{But } \Sigma\left(\frac{w}{g} \cdot v \cdot r\right) = \Sigma\left(\frac{w}{g} \cdot \omega r \cdot r\right) = \Sigma\left(\frac{w}{g}\omega r^2\right)$$

$$= \omega\Sigma\left(\frac{w}{g}r^2\right) = \mathrm{I} \cdot \omega$$

or the angular momentum is equal to the moment of inertia (or angular inertia) multiplied by the angular velocity.

Suppose the velocity of P increases from v_1 to v_2, the angular velocity increasing from ω_1 to ω_2, the change of angular momentum is—

$$\Sigma\left(\frac{w}{g}rv_2\right) - \Sigma\left(\frac{w}{g}rv_1\right) = \Sigma\left\{\frac{w}{g}r(v_2 - v_1)\right\}$$

If the change occupies a time t seconds, the mean rate of change of angular momentum of the whole body is—

$$\Sigma\left(\frac{w}{g} \cdot r \cdot \frac{v_2 - v_1}{t}\right) = \Sigma\left(\frac{w}{g} \cdot f \cdot r\right) = \Sigma(\mathrm{F}r)$$

where f is the average acceleration of P during the time t, and $\frac{w}{g}f$ or F is the average effective accelerating force on the particle at P, acting always in its direction of motion, *i.e.* acting always tangentially to the circular path of P (see Art. 40).

Also Σ (F . r) is the average total moment of the effective or net forces acting on the various particles of the body or the average effective torque on the body.

If these average accelerations and forces be estimated over indefinitely small intervals of time, the same relations are true, and ultimately the rate of change of angular momentum is equal to the moment of the forces producing the change, so that—

rate of change of $\mathrm{I}\omega = \Sigma(\mathrm{F}r) = \mathrm{M}$

= total algebraic moment of effective forces, or effective torque

Also—

$$\text{rate of change of } I\omega = I \times \text{rate of change of } \omega$$

or I . a, where a is the angular acceleration or rate of change of angular velocity. Hence—

$$\Sigma(Fr) = M = I a$$

a result otherwise obtained for the special case of uniform acceleration in (2), Art. 146.

Problems can often be solved alternately from equation (1) or equation (2) (Art. 146), just as in the case of linear motion the equation of energy (Art. 60) or that of force (Art. 51) can be used (Art. 60).

Example 1.—A flywheel weighing 200 lbs. is carried on a spindle 2·5 inches diameter. A string is wrapped round the spindle, to which one end is loosely attached. The other end of the string carries a weight of 40 lbs., 4 lbs. of which is necessary to overcome the friction (assumed constant) between the spindle and its bearings. Starting from rest, the weight, pulling the flywheel round, falls vertically through 3 feet in 7 seconds. Find the moment of inertia and radius of gyration of the flywheel.

The average velocity of the falling weight is $\frac{3}{7}$ foot per second, and since under a uniform force the acceleration is uniform, the maximum velocity is $2 \times \frac{3}{7}$ or $\frac{6}{7}$ foot per second.

The net work done by the falling weight, *i.e.* the whole work done minus that spent in overcoming friction, is—

$$(40 - 4)3 \text{ foot-lbs.} = 108 \text{ foot-lbs.}$$

The kinetic energy of the falling weight is—

$$\frac{1}{2} \cdot \frac{40}{32\cdot2} \cdot (\tfrac{6}{7})^2 = 0\cdot456 \text{ foot-lb.}$$

If I = moment of inertia of the flywheel, and ω = its angular velocity in radians per second. By the principle of work (Art. 61)—

$$\tfrac{1}{2}I\omega^2 + 0\cdot456 = 108 \text{ foot-lbs.}$$
$$\tfrac{1}{2}I\omega^2 = 108 - 0\cdot456 = 107\cdot544 \text{ foot-lbs.}$$

The maximum angular velocity ω is equal to the maximum linear velocity of the string in feet per second divided by the radius of the spindle in feet, or—

$$\omega = \frac{6}{7} \div \left(\frac{1\cdot25}{12}\right) = \frac{6}{7} \times \frac{12}{1\cdot25} = \frac{72}{8\cdot75}$$
$$= 8\cdot22 \text{ radians per second}$$

therefore $\frac{1}{2}\mathrm{I} \times (8\cdot22)^2 = 107\cdot544$

$$\mathrm{I} = \frac{107\cdot544 \times 2}{(8\cdot22)^2} = \frac{215\cdot1}{67\cdot6} = 3\cdot18 \text{ units}$$

And if k = radius of gyration in feet, since the wheel weighs 200 lbs.—

$$\frac{200}{32\cdot2} \cdot k^2 = 3\cdot18$$
$$k^2 = 0\cdot518 \text{ (foot)}^2$$
$$k = 0\cdot716 \text{ foot or } 8\cdot6 \text{ inches}$$

Example 2.—An engine in starting exerts on the crank-shaft for one minute a constant turning moment of 1000 lb.-feet, and there is a uniform moment resisting motion, of 800 lb.-feet. The flywheel has a radius of gyration of 5 feet and weighs 2000 lbs. Neglecting the inertia of all parts except the flywheel, what speed will the engine attain during one minute?

(1) Considering the rate of change of angular momentum—

The effective turning moment is $1000 - 800 = 200$ lb.-feet

The moment of inertia of the flywheel is $\frac{2000}{32\cdot3} \times 5^2 = 1552$ units

Hence if α = angular acceleration in radians per second per second

$$200 = 1552\alpha \text{ (Art. 146 (2))}$$
$$\alpha = \frac{200}{1552} = 0\cdot1287 \text{ radian per second per second}$$

And the angular velocity attained in one minute is—

$$60 \times 0\cdot1287 = 7\cdot74 \text{ radians per second}$$
$$\text{or } \frac{7\cdot74 \times 60}{2\pi} = 74 \text{ revolutions per minute}$$

(2) Alternatively from considerations of energy.

If ω = angular velocity acquired

$\frac{\omega}{2}$ = mean angular velocity

Total angle turned through in one minute $\Big\} = 60 \times \frac{\omega}{2} = 30\omega$ radians

Net work done in one minute $= 200 \times 30\omega$ foot-lbs.

$$200 \times 30\omega = \tfrac{1}{2}\mathrm{I}\omega^2$$
$$6000\omega = \tfrac{1}{2} \cdot 1552 \cdot \omega^2$$
$$\omega = \frac{12000}{1552} = 7\cdot74 \text{ radians per second as before}$$

Example 3.—A thin straight rod of uniform material, 4·5 feet long, is hinged at one end so that it can turn in a vertical plane. It is placed in a horizontal position, and then released. Find the velocity of the free end (1) when it has described an angle of 30°, (2) when it is vertical.

(1) After describing 30° the centre of gravity G (Fig. 173), which is then at G_1, has fallen a vertical distance ON.

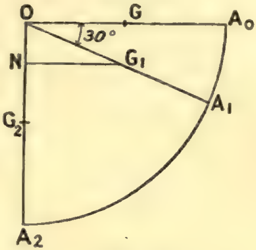


Fig. 173.

$$ON = OG_1 \cos 60° = \tfrac{1}{2}OG_1 = \tfrac{1}{2} \times 2·25$$
$$= 1·125 \text{ feet}$$

If W is the weight of the rod in pounds, the work done by gravitation is—

$$W \times 1·125 \text{ foot-lbs.}$$

The moment of inertia of the rod $\left(\frac{W}{g}k^2\right)$ is—

$$\frac{W}{g} \cdot \frac{(4·5)^2}{3} = \frac{27}{4} \cdot \frac{W}{g}$$

If ω_1 is the angular velocity of the rod, since the kinetic energy of the rod must be 1·125W foot-lbs.—

$$\tfrac{1}{2} \cdot \tfrac{27}{4} \cdot \frac{W}{g}\omega_1^2 = 1·125W$$
$$\omega_1^2 = \tfrac{9}{8} \times \tfrac{8}{27} \times 32·2 = 10·73$$
$$\omega = 3·28 \text{ radians per second}$$

the velocity of A_0 in position A_1 is then—

$$3·28 \times 4·5 = 14·74 \text{ feet per second}$$

(2) In describing 90° G falls 2·25 feet, and the kinetic energy is then 2·25W foot-lbs.

And if ω_2 is the angular velocity of the rod—

$$\tfrac{1}{2} \cdot \tfrac{27}{4} \cdot \frac{W}{g} \cdot \omega_2^2 = 2·25W$$
$$\omega_2^2 = \tfrac{9}{4} \times \tfrac{8}{27} \times 32·2 = 21·47$$
$$\omega_2 = 4·63 \text{ radians per second}$$

and the velocity of A_0 in the position A_2 is—

$$4·63 \times 4·5 = 20·82 \text{ feet per second}$$

148. Compound Pendulum.—In Art. 71 the motion of a "simple pendulum" was investigated, and it was stated that such a pendulum was only approximated to by any actual pendulum. We now proceed to find the simple pendulum equivalent (in period) to an actual pendulum.

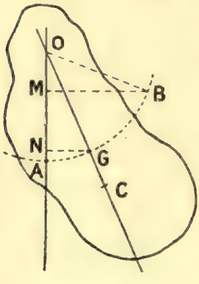


FIG. 174.

Let a body be suspended by means of a horizontal axis O (Fig. 174) perpendicular to the figure and passing through the body. Let G be the c.g. of the body in any position, and let OG make any angle θ with the vertical plane (OA) through O.

Suppose that the body has been raised to such a position that G was at B, and then released. Let the angle $A\hat{O}B$ be ϕ, and $OG = OB = OA = h$.

The body oscillating about the horizontal axis O constitutes a pendulum.

Let l = length of the simple equivalent pendulum (Art. 71);
I = the moment of inertia of the pendulum about the axis O;
k_0 = radius of gyration about O;
k_G = radius of gyration about a parallel axis through G.

Let W be the weight of the pendulum, and let M and N be the points in which horizontal lines through B and G respectively cut OA.

When G has fallen from B to G, the work done is—

$$\begin{aligned} W \times MN &= W(ON - OM) = W(h \cos \theta - h \cos \phi) \\ &= Wh(\cos \theta - \cos \phi) \end{aligned}$$

Let the angular velocity of the pendulum in this position be ω, then its kinetic energy is $\frac{1}{2}I\omega^2$ (Art. 144), and by the principle of work (Art. 61), if there are no resistances to motion the kinetic energy is equal to the work done, or—

$$\tfrac{1}{2}I\omega^2 = Wh(\cos \theta - \cos \phi)$$

and therefore—

$$\omega^2 = \frac{2Wh}{I}(\cos \theta - \cos \phi) \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if a particle (Fig. 175) be attached to a point O′ by a flexible thread of length l, and be released from a position B′ such that $B'\hat{O}'A = \phi$, O′A being vertical, its velocity v when passing G′ such that $G'\hat{O}'A = \theta$ is given by—

$$v^2 = 2g \,.\, M'N' = 2gl(\cos\theta - \cos\phi)$$

and its angular velocity ω about O′ being $\frac{v}{l}$—

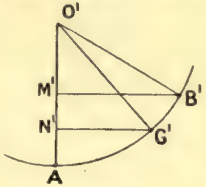


FIG. 175.

$$\omega^2 = \frac{2g}{l}(\cos\theta - \cos\phi) \;. \quad . \quad . \quad . \quad . \quad (2)$$

The angular velocity of a particle (or of a simple pendulum) given by equation (2) is the same as that of G (Fig. 174) given by equation (1), provided—

$$\frac{g}{l} = \frac{Wh}{I} = \frac{Wh \,.\, g}{Wk_0^2}$$

i.e. provided—

$$l = \frac{k_0^2}{h}$$

This length $\frac{k_0^2}{h}$ is then the length of the simple pendulum equivalent to that in Fig. 174, for since the velocity is the same at *any* angular position for the simple pendulum of length l and the actual pendulum, their times of oscillation must be the same. Also, since—

$$k_0^2 = k_G^2 + h^2 \text{ (Art. 137 (2))}$$

$$l = \frac{k_G^2 + h^2}{h} = \frac{k_G^2}{h} + h$$

The point C (Fig. 174), distant $\frac{k_G^2}{h} + h$ from O, and in the line OG is called the "*centre of oscillation.*" The expression $\frac{k_G^2}{h} + h$ shows that it is at a distance $\frac{k_G^2}{h}$ beyond G from O. A particle placed at C would oscillate in the same period about O as does the *compound pendulum* of Fig. 174.

Example.—A flywheel having a radius of gyration of 3·25 feet is balanced upon a knife-edge parallel to the axis of the wheel and inside the rim at a distance of 3 feet from the axis of the wheel. If the wheel is slightly displaced in its own plane, find its period of oscillation about the knife-edge.

The length of the simple equivalent pendulum is—

$$3 + \frac{(3{\cdot}25)^2}{3} = 3 + 3{\cdot}5208 = 6{\cdot}5208 \text{ feet}$$

Hence the period is $2\pi\sqrt{\dfrac{6{\cdot}2508}{32{\cdot}2}} = 2{\cdot}76$ seconds

149. The laws of rotation of a body about an axis may be stated in the same way as Newton's laws of motion as follows :—

Law 1. A rigid body constrained to rotate about an axis continues to rotate about that axis with constant angular velocity except in so far as it may be compelled to change that motion by forces having a moment about that axis.

Law 2. The rate of change of angular momentum is proportional to the moment of the applied forces, or torque about the axis. With a suitable choice of units, the rate of change of angular momentum is *equal* to the moment of the applied forces, or torque about the axis.

Law 3. If a body A exerts a twisting moment or torque about a given axis on a body B, then B exerts an equal and opposite moment or torque about that axis on the body A.

150. Torsional Simple Harmonic Motion.—If a rigid body receives an angular displacement about an axis, and the moment of the forces acting on it tending to restore equilibrium is proportional to the angular displacement, then the body executes a rotary vibration of a simple harmonic kind. Such a restoring moment is exerted when a body which is suspended by an elastic wire or rod receives an angular displacement about the axis of suspension not exceeding a certain limit.

Let M = restoring moment or torque in lb.-feet per radian of twist;

I = moment of inertia of the body about the axis of suspension in engineer's units;

μ = angular acceleration of the body in radians per second per second per radian of twist.

$$\text{Then } M = I \cdot \mu \text{ (Art. 147)}$$
$$\text{or } \mu = \frac{M}{I}$$

Then, following exactly the same method as in Art. 68, if Q (Fig. 176) rotates uniformly with angular velocity $\sqrt{\mu}$ in a circle centred at O and of radius OA, which represents to scale the greatest angular displacement of the body, and P is the projection of Q on OA, then P moves in the same way as a point distant from O by a length representing the angular displacement θ, at any instant to the same scale that OA represents the extreme displacement. The whole argument of Art. 68 need not be repeated here, but the results are—

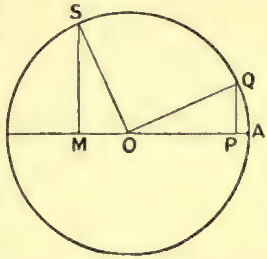


FIG 176.

Angular velocity for an angular displacement θ, represented by OM, is $\sqrt{\mu}\sqrt{\phi^2 - \theta^2}$.

Angular acceleration for an angular displacement θ, represented by PO, is $\sqrt{\mu} \cdot \theta$.

$$T = \text{time of complete vibration} = \frac{2\pi}{\sqrt{\mu}} \text{ seconds}$$

or, since—

$$\mu = \frac{M}{I}$$
$$T = 2\pi\sqrt{\frac{I}{M}}$$

Example.—A metal disc is 10 inches diameter and weighs 6 lbs. It is suspended from its centre by a vertical wire so that its plane is horizontal, and then twisted. When released, how many oscillations will it make per minute if the rigidity of the suspension wire is such that a twisting moment of 1 lb.-foot causes an angular deflection of 10°?

$$\left.\begin{array}{l}\text{The twisting moment per radian twist is} \\ 1 \div \left(\frac{\pi}{180} \times 10\right)\end{array}\right\} = 5{\cdot}73 \text{ lb.-feet}$$

The square of radius of gyration is $\frac{1}{2}(\frac{5}{12})^2 = 0{\cdot}0862$ (foot)^2

The moment of inertia is $\frac{6}{32 \cdot 2} \times 0 \cdot 0862 = 0 \cdot 01615$ unit

Hence the time of vibration is $2\pi\sqrt{\frac{I}{M}} = 2\pi\sqrt{\frac{0 \cdot 01615}{5 \cdot 73}}$

$= 0 \cdot 337$ second

The number of vibrations per minute is then $\left.\frac{60}{0 \cdot 337}\right\} = 178$

151. It is evident, from Articles 144 to 150, that the rotation of a rigid body about an axis bears a close analogy to the linear motion of a body considered in Chapters I. to IV.

Some comparisons are tabulated below.

Linear.	Angular or Rotational.
Mass or inertia, $\frac{w}{g}$ or m.	Moment of inertia, I.
Length, l.	Angular displacement, θ.
Velocity, v.	Angular velocity, ω.
Acceleration, f.	Angular acceleration, α.
Force, F.	Moment of force, or torque, M.
Momentum, $\frac{w}{g} \cdot v$ or mv.	Angular momentum, $I \cdot \omega$.
Average velocity, $\frac{l}{t}$.	Average angular velocity, $\frac{\theta}{t}$.
Average acceleration, $\frac{v_1 - v_2}{t}$.	Average angular acceleration, $\frac{\omega_1 - \omega_2}{t}$
Average force, $\frac{w}{g} \cdot \frac{(v_1 - v_2)}{t}$ or $\frac{m(v_1 - v_2)}{t}$	Average moment or torque, $\frac{I(\omega_1 - \omega_2)}{t}$
Work of constant force, $F \cdot l$.	Work of constant torque, $M \cdot \theta$.
Kinetic energy, $\frac{1}{2}\frac{w}{g}v^2$ or $\frac{1}{2}mv^2$.	Kinetic energy, $\frac{1}{2}I\omega^2$.
Period of simple vibration, $2\pi\sqrt{\frac{m}{e}}$ or $2\pi\sqrt{\frac{w}{eg}}$, where e = force per unit displacement.	Period of simple vibration, $2\pi\sqrt{\frac{I}{M}}$, where M = torque per radian displacement.

The quantities stated as average values have similar meanings when the averages are reckoned over indefinitely small intervals of time, or, in other words, they have corresponding limiting values.

152. Kinetic Energy of a Rolling Body.—We shall limit ourselves to the case of a solid of revolution rolling along a plane. The c.g. of the solid will then be in the axis of revolution about which the solid will rotate as it rolls. Let R be the extreme radius of the body at which rolling contact with the plane takes place (Fig. 177); let the centre O be moving parallel to the plane with a velocity V. Then any point P on the outside circumference of the body is moving with a velocity V relative to O, the angular velocity of P and of the whole body about O being $\frac{V}{R}$, or say ω radians per second.

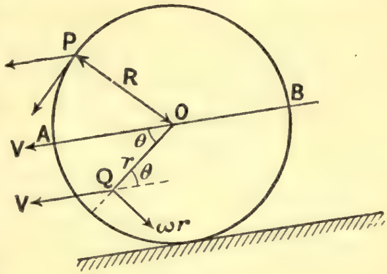


FIG. 177.

Consider the kinetic energy of a particle weighing w lbs. at Q, distant OQ or r from the axis of the body. Let OQ make an angle $Q\hat{O}A = \theta$ with OA, the direction of motion of O. Then the velocity v of Q is the resultant of a velocity V parallel to OA, and a velocity ωr perpendicular to OQ, and is such that—

$$v^2 = (\omega r)^2 + V^2 + 2\omega r \,.\, V \,.\, \cos(90 + \theta)$$

Hence the kinetic energy of the particle is—

$$\frac{1}{2}\frac{w}{g}(\omega^2 r^2 + V^2 - 2\omega r V \sin\theta)$$

The total kinetic energy of the body is then—

$$\Sigma\left(\frac{w}{2g}v^2\right) = \Sigma\left\{\frac{w}{2g}(\omega^2 r^2 + V^2 - 2\omega r V \sin\theta)\right\}$$

$$= \frac{1}{2}\omega^2\Sigma\left(\frac{w}{g}r^2\right) + \frac{V^2}{2g}\Sigma(w) - \frac{2\omega V}{2g}\Sigma(w \,.\, r\sin\theta)$$

Now, $\Sigma(wr \,.\, \sin\theta) = 0$ (Art. 113 (3))

and $\Sigma\left(\frac{w}{g} \,.\, r^2\right) = \mathrm{I}$, the moment of inertia of the solid about the axis O

hence $\Sigma\left(\frac{w}{2g} \,.\, v^2\right) = \frac{1}{2}\mathrm{I}\omega^2 + \frac{1}{2}\frac{\mathrm{W}}{g}\mathrm{V}^2$

= kinetic energy of rotation about O + kinetic energy of an equal weight moving with the linear velocity of the axis.

This may also be written—

$$\tfrac{1}{2}\frac{\mathrm{W}}{g}(k^2 + \mathrm{R}^2)\omega^2, \text{ or } \tfrac{1}{2}\frac{\mathrm{W}}{g}\mathrm{V}^2\left(1 + \frac{k^2}{\mathrm{R}^2}\right)$$

where k is the radius of gyration about the axis O. The kinetic energy $\frac{\mathrm{W}}{2g}\mathrm{V}^2\left(1 + \frac{k^2}{\mathrm{R}^2}\right)$ is then the same as that of a weight $\mathrm{W}\left(1 + \frac{k^2}{\mathrm{R}^2}\right)$ moving with a velocity V of pure translation, *i.e.* without rotation.

In the case of a body rolling down a plane inclined θ to the horizontal (Fig. 178), using the same notation as in the previous case, the component force of gravity through O and parallel to the direction of motion down the plane is $\frac{\mathrm{W}}{g} \,.\, \sin\theta$. In rolling a distance s down the plane, the work done is $\mathrm{W} \sin\theta \,.\, s$. Hence the kinetic energy stored after the distance s is—

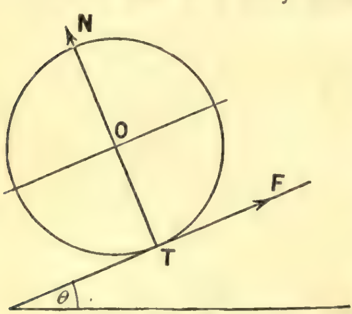


Fig. 178.

$$\tfrac{1}{2}\frac{\mathrm{W}}{g}\mathrm{V}^2\left(1 + \frac{k^2}{\mathrm{R}^2}\right) = \mathrm{W}\sin\theta \,.\, s \text{ (Art. 61)}$$

$$\text{or } \mathrm{V}^2 = 2sg\sin\theta\frac{\mathrm{R}^2}{\mathrm{R}^2 + k^2}$$

This is the velocity which a body would attain in moving

without rotation a distance s from rest under an acceleration $g \sin \theta \frac{R^2}{R^2 + k^2}$. Hence the effect of rolling instead of sliding down the plane is to decrease the linear acceleration and linear velocity attained by the axis in a given *time* in the ratio $\frac{R^2}{R^2 + k^2}$ (see Art. 28).

We may alternatively obtain this result as follows: Resolving the reaction of the (rough) plane on the body at T into components N and F, normal to the plane and along it respectively, the net force acting *down* the plane on the body is W sin θ − F; and if α = angular acceleration of the body about O, and f = linear acceleration down the plane—

$$\alpha = \frac{f}{R}$$

$$\text{But } I\alpha = FR \text{ (Art. 146 (2))}$$

F being the only force which has any moment about O;

$$\text{hence } F = \frac{I\alpha}{R} = \frac{If}{R^2}$$

and the force acting down the plane is $W \sin \theta - \frac{If}{R^2}$.

$$\text{Hence } f = \frac{\text{force acting down the plane}}{\text{mass of body}} = \left(W \sin \theta - \frac{If}{R^2}\right) \div \frac{W}{g}$$

$$f = g \sin \theta - f\frac{k^2}{R^2}$$

$$\text{or } f = g \sin \theta \times \frac{R^2}{R^2 + k^2}$$

Example.—A solid disc rolls down a plane inclined 30° to the horizontal. How far will it move down the plane in 20 seconds from rest? What is then the velocity of its centre, and if it weighs 10 lbs., how much kinetic energy has it?

The acceleration of the disc will be—

$$32{\cdot}2 \times \sin 30^\circ \times \frac{R^2}{R^2 + \frac{R^2}{2}} = 32{\cdot}2 \times \tfrac{1}{2} \times \tfrac{2}{3}$$

$$= 10{\cdot}73 \text{ feet per second per second}$$

In 20 seconds it will acquire a velocity of—

$$20 \times 10{\cdot}73 = 214{\cdot}6 \text{ feet per second}$$

Its average velocity throughout this time will be—

$$\frac{214{\cdot}6}{2} = 107{\cdot}3 \text{ feet per second}$$

It will then move—

$$107{\cdot}3 \times 20 = 2146 \text{ feet}$$

corresponding to a vertical fall of 2146 sin 30° or 1073 feet.

The kinetic energy will be equal to the work done on it in falling 1073 feet, *i.e.* 1073 × 10 = 10,730 foot-lbs.

Examples XVIII.

1. What is the moment of inertia in engineer's units of a flywheel which stores 200,000 foot-lbs. of kinetic energy when rotating 100 times per minute?

2. A flywheel requires 20,000 foot-lbs. of work to be done upon it to increase its velocity from 68 to 70 rotations per minute. What is its moment of inertia in engineer's units?

3. A flywheel, the weight of which is 2000 lbs., has a radius of gyration of 3·22 feet. It is carried on a shaft 3 inches diameter, at the circumference of which a constant tangential force of 50 lbs. opposes the rotation of the wheel. If the wheel is rotating 60 times per minute, how long will it take to come to rest, and how many rotations will it make in doing so?

4. A wheel 6 feet diameter has a moment of inertia of 600 units, and is turning at a rate of 50 rotations per minute. What opposing force applied tangentially at the rim of the wheel will bring it to rest in one minute?

5. A flywheel weighing 1·5 tons has a radius of gyration of 4 feet. If it attains a speed of 80 rotations per minute in 40 seconds, find the mean effective torque exerted upon it in pound-feet?

6. A weight of 40 lbs. attached to a cord which is wrapped round the 2-inch spindle of a flywheel descends, and thereby causes the wheel to rotate. If the weight descends 6 feet in 10 seconds, and the friction of the bearing is equivalent to a force of 3 lbs. at the circumference of the spindle, find the moment of inertia of the flywheel. If it weighs 212 lbs., what is its radius of gyration?

7. If the weight in Question 6, after descending 6 feet, is suddenly released, how many rotations will the wheel make before coming to rest?

8. A flywheel weighing 250 lbs. is mounted on a spindle 2·5 inches

diameter, and is caused to rotate by a falling weight of 50 lbs. attached to a string wrapped round the spindle. After falling 5 feet in 8 seconds, the weight is detached, and the wheel subsequently makes 100 rotations before coming to rest. Assuming the tangential frictional resisting force at the circumference of the axle to be constant throughout the accelerating and stopping periods, find the radius of gyration of the wheel.

9. A rod is hinged at one end so that it can turn in a vertical plane about the hinge. The rod is turned into a position of unstable equilibrium vertically above the hinge and then released. Find the velocity of the end of the rod (1) when it is horizontal; (2) when passing through its lowest position, if the rod is 5 feet long and of uniform small section throughout.

10. A circular cylinder, 3 feet long and 9 inches diameter, is hinged about an axis which coincides with the diameter of one of the circular ends. The axis of the cylinder is turned into a horizontal position, and then the cylinder is released. Find the velocity of the free end of the axis (1) after it has described an angle of 50°, (2) when the axis is passing through its vertical position.

11. A flywheel weighs 5 tons, and the internal diameter of its rim is 6 feet. When the inside of the rim is supported upon a knife-edge passing through the spokes and parallel to its axis, the whole makes, if disturbed, 21 complete oscillations per minute. Find the radius of gyration of the wheel about its axis, and the moment of inertia about that axis.

12. A cylindrical bar, 18 inches long and 3 inches diameter, is suspended from an axis through a diameter of one end. If slightly disturbed from its position of stable equilibrium, how many oscillations per minute will it make?

13. A piece of metal is suspended by a vertical wire which passes through the centre of gravity of the metal. A twist of 8·5° is produced per pound-foot of twisting moment applied to the wire, and when the metal is released after giving it a small twist, it makes 150 complete oscillations a minute. Find the moment of inertia of the piece of metal in engineer's or gravitational units.

14. A flywheel weighing 3 tons is fastened to one end of a shaft, the other end of which is fixed, and the torsional rigidity of which is such that it twists 0·4° per ton-foot of twisting moment applied to the flywheel. If the radius of gyration of the flywheel and shaft combined is 3 feet, find the number of torsional vibrations per minute which the wheel would make if slightly twisted and then released.

15. The weight of a waggon is 2 tons, of which the wheels weigh $\frac{1}{4}$ ton. The diameter of the wheels is 2 feet, and the radius of gyration 0·9 foot. Find the total kinetic energy of the waggon when travelling at 40 miles per hour, in foot-tons.

16. A cylinder is placed on a plane inclined 15° to the horizontal, and is allowed to roll down with its axis horizontal. Find its velocity after it has traversed 25 feet.

17. A solid sphere rolls down a plane inclined α to the horizontal. Find its acceleration. (NOTE.—The square of the radius of gyration of a sphere of radius R is $\frac{2}{5}R^2$.)

18. A motor car weighs W lbs., including four wheels, each of which weigh w lbs. The radius of each wheel is a feet, and the radius of gyration about the axis is k feet. Find the total kinetic energy of the car when moving at v feet per second.

CHAPTER X

ELEMENTS OF GRAPHICAL STATICS

153. IN Chapter VI. we considered and stated the conditions of equilibrium of rigid bodies, limiting ourselves to those subject to forces in one plane only. In the case of systems of concurrent forces in equilibrium (Chapter V.), we solved problems alternatively by analytical methods of resolution along two rectangular axes, or by means of drawing vector polygons of forces to scale. We now proceed to apply the vector methods to a few simple systems of non-concurrent forces, such as were considered from the analytical point of view in Chapter VI., and to deduce the vector conditions of equilibrium.

When statical problems are solved by graphical methods, it is usually necessary to first draw out a diagram showing correctly the inclinations of the lines of action of the various known forces to one another, and, to some scale, their relative positions. Such a diagram is called a diagram of positions, or *space diagram;* this is not to be confused with the vector diagram of forces, which gives magnitudes and directions, but not positions of forces.

154. Bows' Notation.—In this notation the lines of action of each force in the *space diagram* are denoted by two letters placed one on each side of its line of action. Thus the spaces rather than the lines or intersections have letters assigned to them, but the limits of a space having a particular letter to denote it may be different for different forces.

The corresponding force in the *vector diagram* has the same two letters at its ends as are given to the spaces separated by

its line of action in the space diagram. We shall use capital letters in the space diagram, and the corresponding small letters to indicate a force in the vector diagram. The notation will be best understood by reference to an example. It is shown in Fig. 179, applied to a space diagram and vector polygon for

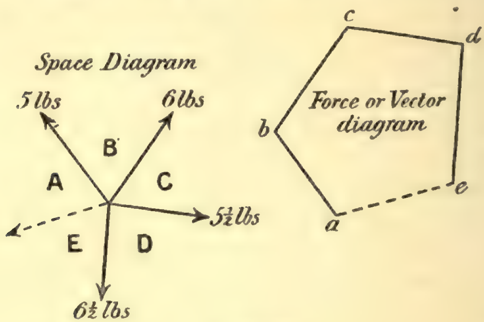


FIG. 179.

five concurrent forces in equilibrium (see Chapter V.). The four forces, AB, BC, CD, DE, of 5 lbs., 6 lbs., $5\frac{1}{2}$ lbs., and $6\frac{1}{2}$ lbs. respectively, being given, the vectors *ab*, *bc*, *cd*, *de* are drawn in succession, of lengths representing to scale these magnitudes and parallel to the lines AB, BC, CD, and DE respectively, the vector *ea*, which scales 5·7 lbs., represents the equilibrant of the four forces, and its position in the space diagram is shown by drawing a line EA parallel to *ea* from the common intersection of AB, BC, CD, and DE. (This is explained in Chapter V., and is given here as an example of the system of lettering only.)

155. The Funicular or Link Polygon.—To find graphically the single resultant or equilibrant of any system of non-concurrent coplanar forces. Let the four forces AB, BC, CD, and DE (Fig. 180) be given completely, *i.e.* their lines of action (directions and positions) and also their magnitudes. First draw a vector *ab* parallel to AB, and representing by its length the given magnitude of the force AB; from *b* draw *bc* parallel to the line BC, and representing the force BC completely. Continuing in this way, as in Art. 73, draw the open

vector or force polygon *abcde*; then, as in the case of concurrent forces, Art. 73, the vector *ae* represents the resultant (or *ea*, the equilibrant) in magnitude and direction. The problem is not yet complete, for the position of the resultant is unknown. In Chapter VI. its position was determined by finding what moment it must have about some fixed point. The graphical method is as follows (the reader is advised to

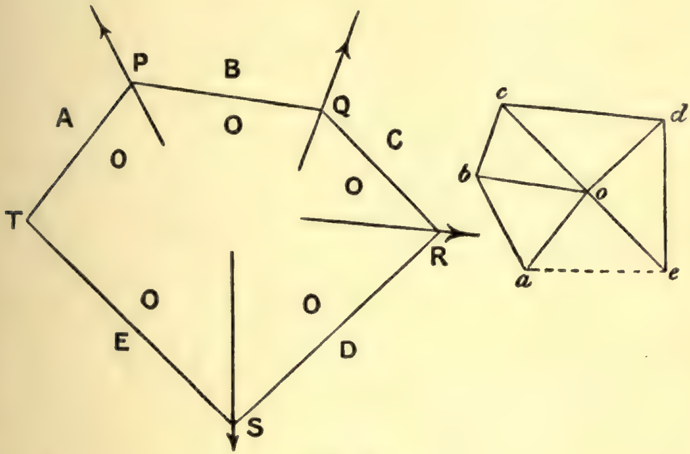


FIG. 180.

draw the figure on a sheet of paper as he reads): Choose any convenient point *o* (called a pole) in or about the vector polygon, and join each vertex *a*, *b*, *c*, *d*, and *e* of the polygon to *o*; then in the space diagram, selecting a point P on the line AB, draw a line PT (which may be called AO) parallel to *ao* across the space A. From P across the space B draw a line BO parallel to *bo* to meet the line BC in Q. From Q draw a line CO parallel to *co* to meet the line CD in R. From R draw a line DO parallel to *do* to meet the line DE in S, and, finally, from S draw a line EO parallel to *eo* to meet the line AO (or PT) in T. Then T, the intersection of AO and EO, is a point in the line of action of EA, the equilibrant, the magnitude and inclination of which were found from the vector *ea*.

Hence the equilibrant EA or the resultant AE is completely determined. The closed polygon PQRST, having its vertices on the lines of action of the forces, is called a *funicular* or *link polygon*. That T must be a point on the line of action of the resultant is evident from the following considerations. Any force may be resolved into two components along any two lines which intersect on its line of action, for it is only necessary for the force to be the geometric sum of the components. (Art. 75). Let each force, AB, BC, CD, and DE, be resolved along the two sides of the funicular polygon which meet on its line of action, viz. AB along TP and QP, BC along PQ and RQ, and so on. The magnitude of the two components is given by the corresponding sides of the triangle of forces in the vector diagram, *e.g.* AB may be replaced by components in the lines AO and BO (or TP and QP), represented in magnitude by the lengths of the vectors *ao* and *ob* respectively, for in vector addition—

$$ao + ob = ab \text{ (Art. 19)}$$

Similarly, CD is replaced by components in the lines CO and OD represented by *co* and *od* respectively. When this process is complete, all the forces AB, BC, CD, and DE are replaced by components, the lines of action of which are the sides TP, PQ, QR, etc., of the funicular polygon. Of these component forces, those in the line PQ or BO are represented by the vectors *ob* and *bo*, and therefore have a resultant nil. Similarly, all the other components balance in pairs, being equal and opposite in the same straight line, except those in the lines TP and TS, represented by *ao* and *oe* respectively. These two have a resultant represented by *ae* (since in vector addition $ao + oe = ae$), which acts through the point of intersection T of their lines of action. Hence finally the resultant of the whole system acts through T, and is represented in magnitude and direction by the line *ae*; the equilibrant is equal and opposite in the same straight line.

156. Conditions of Equilibrium.—If we include the equilibrant EA (Fig. 180, Art. 155) with the other four forces, we have five forces in equilibrium, and (1) the force or vector

polygon *abcde* is closed; and (2) the funicular polygon PQRST is a closed figure. Further, if the force polygon is *not* closed, the system reduces to a single resultant, which may be found by the method just described (Art. 155).

It may happen that the force polygon is a closed figure, and that the funicular polygon is not. Take, for example, a diagram (Fig. 181) similar to the previous one, and let the

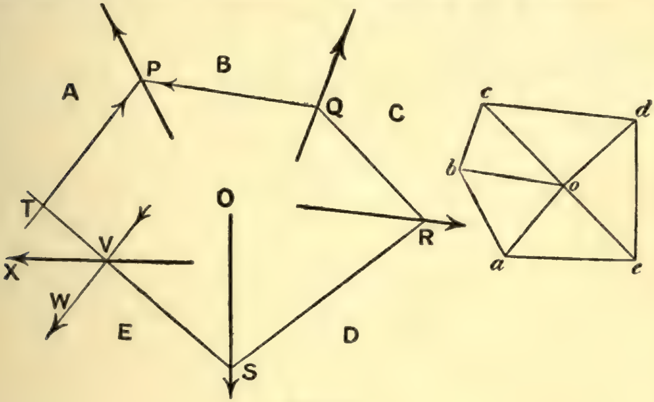


FIG. 181.

forces of the system be AB, BC, CD, DE, and EA, the force EA *not* passing through the point T found in Fig. 180, but through a point V (Fig. 181), in the line TS. If we draw a line, VW, parallel to *oa* through V, it will *not* intersect the line TP parallel to *ao*, for TP and VW are then parallel. Replacing the original forces by components, the lines of action of which are in the sides of the funicular polygon, we are left with two parallel unbalanced components represented by *ao* and *oa* in the lines TP and VW respectively. These form a couple (Art. 91), and such a system is not in equilibrium nor reducible to a single resultant. The magnitude of the couple is equal to the component represented by *oa* multiplied by the length represented by the perpendicular distance between the lines TP and VW. A little consideration will show that it is also equal to the force EA represented by *ea*, multiplied by the distance represented by the perpendicular from T on the

line VX. Or the resultant of the forces in the lines AB, BC, CD, and DE is a force represented by *ae* acting through the point T; this with the force through V, and represented by *ea*, forms a couple.

Hence, for equilibrium it is essential that (1) the polygon of forces is a closed figure; (2) that the funicular polygon is a closed figure.

Compare these with the equivalent statements of the analytical conditions in Art. 96.

Choice of Pole.—In drawing the funicular polygon, the pole *o* (Figs. 180 and 181) was chosen in any arbitrary position, and the first side of the funicular polygon was drawn from *any* point P in the line AB. If the side AO had been drawn from any point in AB other than P, the funicular polygon would have been a similar and similarly situated figure to PQRST.

The choice of a different pole would give a different shaped funicular polygon, but the points in the line of action of the unknown equilibrant obtained from the use of different poles would all lie in a straight line. This may be best appreciated by trial.

Note that in any polygon the sides are each parallel to a line radiating from the corresponding pole.

157. Funicular Polygon for Parallel Forces. — To find the resultant of several parallel forces, we proceed exactly as in the previous case, but the force polygon has its sides all in the same straight line; it is "closed" if, after drawing the various vectors, the last terminates at the starting-point of the first. The vector polygon does not enclose a space, but may be looked upon as a polygon with overlapping sides.

Let the parallel forces (Fig. 182) be AB, BC, CD, and DE of given magnitudes. Set off the vector *ab* in the vector polygon parallel to the line AB, and representing by its length the magnitude of the force in the line AB. And from *b* set off *bc* parallel to the line BC, and representing by its length the magnitude of the force in the line BC. Then *bc* is evidently in the same straight line as *ab*, since AB and BC are parallel. Similarly the vectors *cd*, *de*, and the resultant *ae* of

the polygon are all in the same straight line. Choose any pole o, and join a, b, c, d, and e to o. Then proceed to put in the funicular polygon in the space diagram as explained in

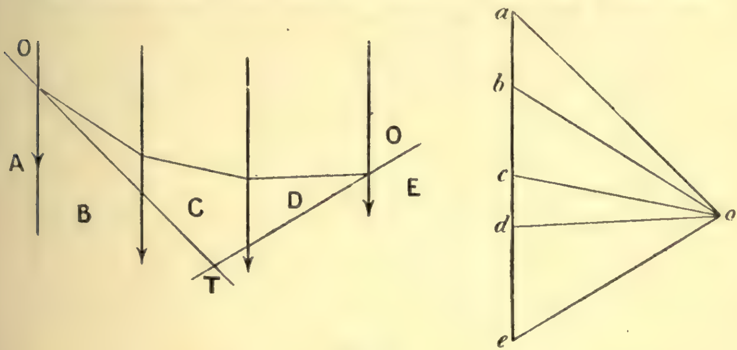


FIG. 182.

Art. 155. The two extreme sides AO and EO intersect in T, and the resultant AE, given in magnitude by the vector ae, acts through this point, and is therefore completely determined.

158. To find Two Equilibrants in Assigned Lines of Action to a System of Parallel Forces.

As a simple example, we may take the vertical reactions

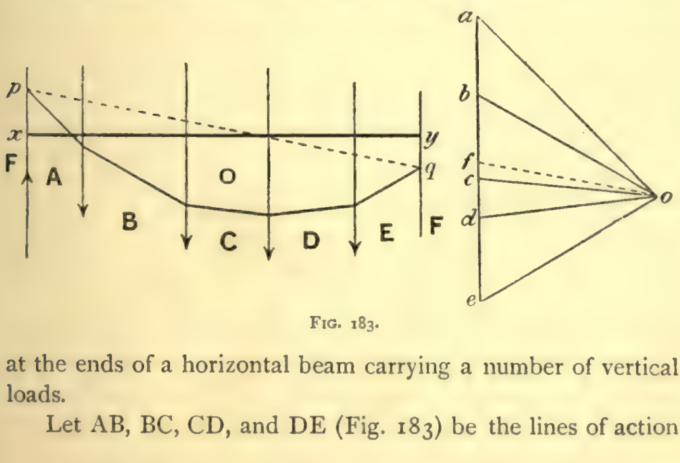


FIG. 183.

at the ends of a horizontal beam carrying a number of vertical loads.

Let AB, BC, CD, and DE (Fig. 183) be the lines of action

of the forces of given magnitudes, being concentrated loads on a beam, xy, supported by vertical forces, EF and FA, at y and x respectively. Choose a pole, o, as before (Arts. 156 and 157), and draw in the funicular polygon with sides AO, BO, CO, DO, and EO respectively parallel to *ao*, *bo*, *co*, *do*, and *eo* in the vector diagram. Let AO meet the line FA (*i.e.* the vertical through x) in p, and let q be the point in which EO meets the line EF (*i.e.* the vertical through y). Join pq, and from o draw a parallel line *of* to meet the line *abcde* in f. The magnitude of the upward reaction or supporting force in the line EF is represented by *ef*, and the other reaction in the line FA is represented by the vector *fa*. This may be proved in the same way as the proposition in Art. 155.

af and *fe* represent the downward pressure of the beam at x and y respectively, while *fa* and *ef* represent the upward forces exerted by the supports at these points.

159. In the case of non-parallel forces two equilibrants can be found—one to have a given line of action, and the other to pass through a given point, *i.e.* to fulfil altogether three conditions (Art. 96).

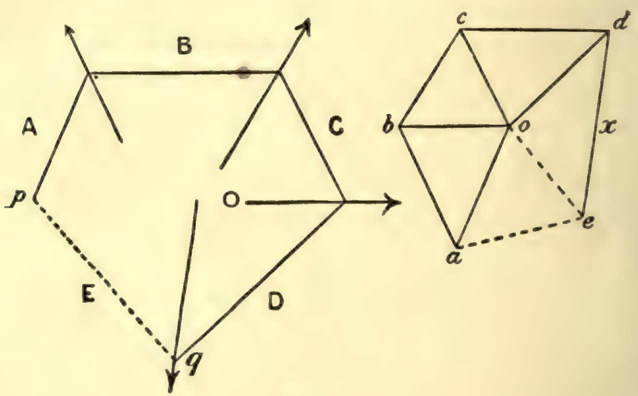


FIG. 184.

Let AB, BC, and CD (Fig. 184) be the lines of action of given forces represented in magnitude by *ab*, *bc*, and *cd* respectively in the vector polygon. Let ED be the line of action of

one equilibrant, and p a point in the line of action of the second. Draw a line, dx, of indefinite length parallel to DE. Choose a pole, o, and draw in the funicular polygon corresponding to it, *but drawing the side AO through the given point p.* Let the last side DO cut ED in q. Then, since the complete funicular polygon is to be a closed figure, join pq. Then the vector oe is found by drawing a line, oe, through o parallel to pq to meet dx in e. The magnitude of the equilibrating force in the line DE is represented by the length de, and the magnitude and direction of the equilibrant EA through p is given by the length and direction of ea.

160. Bending Moment and Shearing Force.—In considering the equilibrium of a rigid body (Chapter VI.), we have hitherto generally only considered the body as a whole. The same conditions of equilibrium must evidently apply to any part of the body we may consider (see Method of Sections, Art. 98). For example, if a beam (Fig. 185) carrying loads W_1, W_2, W_3, W_4, and W_5, as shown, be ideally divided into two

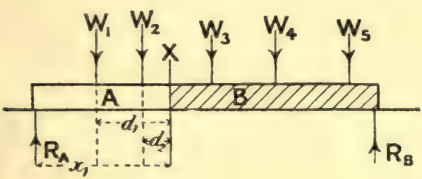


FIG. 185.

parts, A and B, by a plane of section at X, perpendicular to the length of the beam, each part, A and B, may be looked upon as a rigid body in equilibrium under the action of forces. The forces acting on the portion A, say, fulfil the conditions of equilibrium (Art. 96), provided we include in them the forces which the portion B exerts on the portion A.

Note that the reaction of A on B is equal and opposite to the action of B on A, so that these internal forces in the beam make no contribution to the net forces or moment acting on the beam as a whole.

For convenience of expression, we shall speak of the beam

as horizontal and the loads and reactions as vertical forces. Let R_A and R_B be the reactions of the supports on the portions A and B respectively.

Considering the equilibrium of the portion A, since the algebraic sum of the vertical forces on A is zero, B must exert on A an *upward* vertical force $W_1 + W_2 - R_A$. This force is called the *shearing force* at the section X, and may be denoted by F_X. Then

$$F_X = W_1 + W_2 - R_A, \text{ or } W_1 + W_2 - R_A - F_X = 0$$

If the sum $W_1 + W_2$ is numerically less than R_A, F_X is negative, *i.e.* acts downwards on A.

The *shearing force* at any section of this horizontal beam is then numerically equal to the algebraic sum of all the vertical forces acting on either side of the section.

Secondly, since the algebraic sum of all the horizontal forces on A is zero, the resultant horizontal force exerted by B on A must be zero, there being no other horizontal force on A. Again, if x_1, d_1, and d_2 are the horizontal distances of R_A, W_1, and W_2 respectively from the section X, since W_1, W_2, and R_A exert on A a clockwise moment in the plane of the figure about any point in the section X, of magnitude—

$$R_A . x_1 - W_1 . d_1 - W_2 . d_2$$

B must exert on A forces which have a *contra-clockwise* moment M_X, say, numerically equal to $R_A . x_1 - W_1d_1 - W_2d_2$, for the algebraic sum of the moments of all the forces on A is zero, *i.e.*—

$$(R_A . x_1 - W_1d_1 - W_2d_2) - M_X = 0$$

$$\text{or } M_X = R_A . x_1 - W_1d_1 - W_2d_2$$

This moment cannot be exerted by the force F_X, which has zero moment in the plane of the figure about any point in the plane X. Hence, since the horizontal forces exerted by B on A have a resultant zero, they must form a couple of contra-clockwise moment, M_X, *i.e.* any pull exerted by B must be accompanied by a push of equal magnitude. This couple M_X is called the moment of resistance of the beam at the section X, and it is *numerically* equal to the algebraic sum of

moments about that section, of all the forces acting to either side of the section. This algebraic sum of the moments about the section, of all the forces acting to either side of the section X, is called the bending moment at the section X.

161. Determination of Bending Moments and Shearing Forces from a Funicular Polygon.—Confining ourselves again to the horizontal beam supported by vertical forces at each end and carrying vertical loads, it is easy to show that the vertical height of the funicular polygon at any distance along the beam is proportional to the bending moment

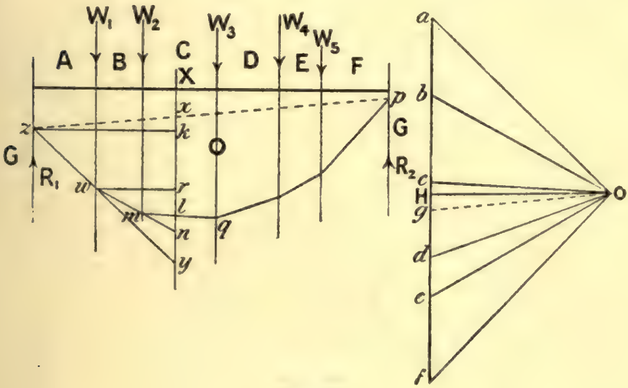


Fig. 186.

at the corresponding section of the beam, and therefore represents it to scale, *e.g.* that xl (Fig. 186) represents the bending moment at the section X.

Let the funicular polygon for any pole o, starting say from z, be drawn as directed in Arts. 155 and 157, og being drawn parallel to zp or GO, the closing line of the funicular, so that R_1, the left-hand reaction, is represented by the vector ga and R_2 by fg, while the loads W_1, W_2, W_3, W_4, and W_5 are represented by the vectors ab, bc, cd, de, and ef respectively. Consider any vertical section, X, of the beam at which the height of bending-moment diagram is xl. Produce xl and the side zw to meet in y. Also produce the side wm of the funicular

polygon to meet xy in n, and let the next side mq of the funicular meet xy in l. The sides zw, wm, and mq (or AO, BO, and CO) are parallel to ao, bo, and co respectively. Draw a horizontal line, zk, through z to meet xy in k, a horizontal line through w to meet xy in r, and a horizontal oH through o in the vector polygon to meet the line $abcdef$ in H. Then in the two triangles xyz and gao there are three sides in either parallel respectively to three sides in the other, hence the triangles are similar, and—

$$\frac{xy}{ag} = \frac{zy}{ao} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also the triangles zky and $o\mathrm{H}a$ are similar, and therefore—

$$\frac{zy}{ao} = \frac{zk}{o\mathrm{H}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence from (1) and (2)—

$$\frac{xy}{ag} = \frac{zk}{o\mathrm{H}}, \text{ or } xy \, . \, o\mathrm{H} = ag \times zk, \text{ or } xy = \frac{ag \, . \, zk}{o\mathrm{H}}$$

Therefore, since ag is proportional to R_1, and zk is equal or proportional to the distance of the line of action of R_1 from X, $ag \, . \, zk$ is proportional to the moment of R_1 about X, and oH being an arbitrarily fixed constant, xy is proportional to the moment of R_1 about X.

Similarly—

$$yn = \frac{ab \, . \, wr}{o\mathrm{H}}$$

and therefore yn represents the moment of W_1 about X to the *same scale* that xy represents the moment of R_1 about X. Similarly, again, nl represents the moment of W_2 about X to the *same scale*.

Finally, the length xl or $(xy - ny - ln)$ represents the algebraic sum of the moments of all the forces to the left of the section X, and therefore represents the bending moment at the section X (Art. 160).

Scales.—If the scale of forces in the vector diagram is—

1 inch to p lbs.

and the scale of distance in the space diagram is—

1 inch to q feet;

and if oH is made h inches long, the scale on which xl represents the bending moment at X is—

1 inch to $p.q.h.$ foot-lbs.

A diagram (Fig. 187) showing the shearing force along the

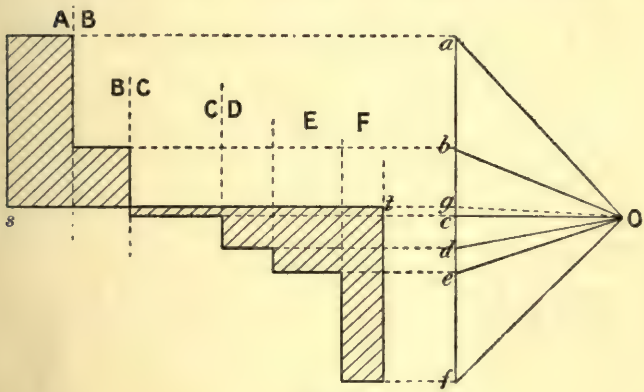


FIG. 187.

length of the beam may be drawn by using a base line, st, of the same length as the beam in the space diagram, and in the horizontal line through g in the force diagram. The shearing force between the end of the beam s and the line AB is constant and equal to R_1, *i.e.* proportional to ga. The height ga may be projected from a by a horizontal line across the space A. A horizontal line drawn through b gives by its height above g the shearing force at all sections of the beam in the space B. Similarly projecting horizontal lines through c, d, e, and f we get a stepped diagram, the height of which from the base line st gives, to the same scale as the vector diagram, the shearing force at every section of the beam.

EXAMPLES XIX.

1. Draw a square lettered continuously PQRS, each side 2 inches long. Forces of 9, 7, and 5 lbs. act in the directions RP, SQ, and QR respectively. Find by means of a funicular polygon the resultant of these three forces. State its magnitude in pounds, its perpendicular distance from P, and its inclination to the direction PQ.

2. Add to the three forces in question 1 a force of 6 lbs. in the direction PQ, and find the resultant as before. Specify it by its magnitude, its distance from P, and its inclination to PQ.

3. A horizontal beam, 15 feet long, resting on supports at its ends, carries concentrated vertical loads of 7, 9, 5, and 8 tons at distances of 3, 8, 12, and 14 feet respectively from the left-hand support. Find graphically the reactions at the two supports.

4. A horizontal rod AB, 13 feet long, is supported by a horizontal hinge perpendicular to AB at A, and by a vertical upward force at B. Four forces of 8, 5, 12, and 17 lbs. act upon the rod, their lines of action cutting AB at 1, 4, 8, and 12 feet respectively from A, their lines of action making angles of 70°, 90°, 120°, and 135° respectively with the direction AB, each estimated in a clockwise direction. Find the pressure exerted on the hinge, state its magnitude, and its inclination to AB.

5. A simply supported beam rests on supports 17 feet apart, and carries loads of 7, 4, 2, and 5 tons at distances of 3, 8, 12, and 14 feet respectively from the left-hand end. Calculate the bending moment at 4, 9, and 11 feet from the left-hand end.

6. Draw a diagram to show the bending moments at all parts of the beam in question 5. State the scales of the diagram, and measure from it the bending moment at 9, 11, 13, and 14 feet from the left-hand support.

7. Calculate the shearing force on a section of the beam in Question 5 at a point 10 feet from the left-hand support; draw a diagram showing the shearing force at every transverse section of the beam, and measure from it the shearing force at 4 and at 13 feet from the left-hand support.

8. A beam of 20-feet span carries a load of 10 tons evenly spread over the length of the beam. Find the bending moment and shearing force at the mid-section and at a section midway between the middle and one end.

162. Equilibrium of Jointed Structures.

Frames.—The name *frame* is given to a structure consisting of a number of bars fastened together by hinged joints; the separate bars are called members of the frame. Such structures are designed to carry loads which are applied mainly at the joints. We shall only consider frames which have just a sufficient number of members to prevent deformation or collapse under the applied loads. Frames having more

members than this requirement are treated in books on Graphical Statics and Theory of Structures. We shall further limit ourselves mainly to frames all the members of which are approximately in the same plane and acted upon by forces all in this same plane and applied at the hinges.

Such a frame is a rigid body, and the forces exerted upon it when in equilibrium must fulfil the conditions stated in Art. 96 and in Art. 156. These "external" forces acting on the frame consist of applied loads and reactions of supports; they can be represented in magnitude and direction by the sides of a closed vector polygon; also their positions are such that an indefinite number of closed funicular polygons can be drawn having their vertices on the lines of action of the external forces. From these two considerations the complete system of external forces can be determined from sufficient data, as in Arts. 155 and 159. The "internal" forces, *i.e.* the forces exerted by the members on the joints, may be determined from the following principle. The pin of each hinged joint is in equilibrium under the action of several forces which are practically coplanar and concurrent. These forces are: the stresses in the members (or the "internal" forces) meeting at the particular joint, and the "external" forces, *i.e.* loads and reactions, if any, which are applied there.

If all the forces, except two internal ones, acting at a given joint are known, then the two which have their lines of action in the two bars can be found by completing an open polygon of forces by lines parallel to those two bars.

If a closed polygon of forces be drawn for each joint in the structure, the stress in every bar will be determined. In order to draw such a polygon for any particular joint, all the concurrent forces acting upon it, except two, must be known, and therefore a start must be made by drawing a polygon for a joint at which some external force, previously determined, acts. Remembering that the forces which any bar exerts on the joints at its two ends are equal and in opposite directions, the drawing of a complete polygon for one joint supplies a means of starting the force polygon for a neighbouring joint for which at least one side is then known. An example of the determination of

the stresses in the members of a simple frame will make this more easily understood.

Fig. 188 shows the principles of the graphical method of finding the stresses or internal forces in the members of a simple frame consisting of five bars, the joints of which have been denoted at (*a*) by 1, 2, 3, and 4. The frame stands in

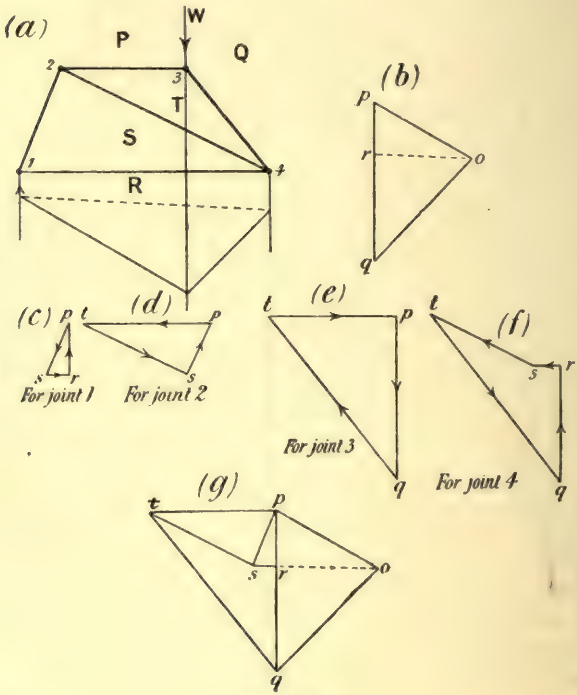


Fig. 188.

the vertical plane, and carries a known vertical load, W, at the joint 3; it rests on supports on the same level at 1 and 4. The force W is denoted in Bow's notation by the letters PQ. The reactions at 1 and 4, named RP and QR respectively, have been found by a funicular polygon corresponding to the vector diagram at (*b*), as described in Art. 158.

Letters S and T have been used for the two remaining spaces. When the upward vertical force RP at the joint 1 is known, the triangle of forces *rps* at (*c*) can be drawn by making *rp* proportional to RP as in (*b*), and completing the triangle by sides parallel to PS and SR (*i.e.* to the bars 12 and 14) respectively. After this triangle has been drawn, one of the three forces acting at the joint 2 is known, viz. SP acting in the bar 12, being equal and opposite to PS in (*c*). Hence the triangle of forces *spt* at (*d*), for the joint 2 can be drawn. Next the triangle *tpq* at (*e*) for joint 3 can be drawn, *tp* and *pq* being known; the line joining *qt* will be found parallel to the bar QT if the previous drawing has been correct; this is a check on the accuracy of the results. Finally, the polygon *qrst* at (*f*) for joint 4 may be drawn, for all four sides are known in magnitude and direction from the previous polygons. The fact that when drawn to their previously found lengths and directions they form a closed polygon, constitutes a check to the correct setting out of the force polygons. The arrow-heads on the sides of the polygons denote the directions of the forces on the particular joint to which the polygon refers.

163. Stress Diagrams.—It is to be noticed in Fig. 188 that in the polygons (*b*), (*c*), (*d*), (*e*), and (*f*), drawn for the external forces on the frame and the forces at the various joints, each side, whether representing an external or internal force, has a line of equal length and the same inclination in some other polygon.

For example, *sr* in (*c*) corresponding to *rs* in (*f*), and *pt* in (*d*) with *tp* in (*e*). The drawing of entirely separate polygons for the forces at each joint is unnecessary; they may all be included in a single figure, such as (*g*), which may be regarded as the previous five polygons superposed, with corresponding sides coinciding. Such a figure is called a stress diagram for the given frame under the given system of external loading. It contains (1) a closed vector polygon for the system of external forces in the frame, (2) closed vector polygons for the (concurrent) forces at each joint of the structure.

As each vector representing the internal force in a member of the frame represents two equal and opposite forces,

arrow-heads on the vectors are useless or misleading, and are omitted.

Distinction between Tension and Compression Members of a Frame.—A member which is in tension is called a "tie," and is subjected by the joints at its ends to a pull tending to lengthen it. The forces which the member exerts on the joints at its ends are equal and opposite *pulls* tending to bring the joints closer together.

A member which is in compression is called a "strut;" it has exerted upon it by the joints at its ends two equal and opposite pushes or thrusts tending to shorten it. The member exerts on the joints at its ends equal and opposite "outward" thrusts tending to force the joints apart.

The question whether a particular member is a "tie" or a "strut" may be decided by finding whether it pulls or thrusts at a joint at either end. This is easily discovered if the direction of any of the forces at that joint is known, since the vector polygon is a closed figure with the last side terminating at the point from which the first was started. *E.g.* to find the kind of stress in the bar 24, or ST (Fig. 188). At joint 4 QR is an upward force; hence the forces in the polygon *qrst* must act in the directions $\overrightarrow{qr}$, $\overrightarrow{rs}$, $\overrightarrow{st}$, and $\overrightarrow{tq}$; hence the force ST in bar 24 acts at joint 4 in the direction *s* to *t*, *i.e.* the bar pulls at joint 4, or the force in ST is a tension. Similarly, the force in bar 23, or PT, acts at joint 3 in a direction *tp*, *i.e.* it pushes at joint 3, or the force in bar 23 is a compressive one.

Another method.—Knowing the direction of the force *rp* at joint 1 (Fig. 188), we know that the forces at joint 1 act in the directions *rp*, *ps*, and *sr*, or the vertices of the vector polygon *rps* lie in the order *r—p—s*.

The corresponding lines RP, PS, and SR in the space diagram are in *clockwise* order round the point 1. This order, clockwise or contra-clockwise (but in this instance clockwise) is the same for every joint in the frame. If it is clockwise for joint 1, it is also clockwise for joint 2. Then the vertices of the vector polygon for joint 2 are to be taken in the cyclic order *s—p—t*, since the lines SP, PT, and TS lie in clockwise

order round the joint 2, *e.g.* the force in bar 23, or PT, is in the direction $\overrightarrow{pt}$, *i.e.* it *thrusts* at joint 2.

This characteristic order of space letters round the joints is a very convenient method of picking out the kind of stress in one member of a complicated frame. Note that it is the characteristic order of space letters round a joint that is constant—*not* the direction of vectors round the various polygons constituting the stress diagram.

164. Warren Girder.—A second example of a simple stress diagram is shown in Fig. 189, viz. that of a common type

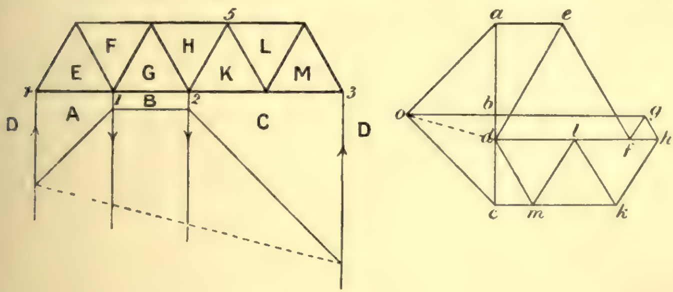


FIG. 189.

of frame called the Warren girder, consisting of a number of bars jointed together as shown, all members generally being of the same lengths, some horizontal, and others inclined 60° to the horizontal.

Two equal loads, AB and BC, have been supposed to act at the joints 1 and 2, and the frame is supported by vertical reactions at 3 and 4, which are found by a funicular polygon. The remaining forces in the bars are found by completing the stress diagram *abc . . . klm*.

Note that the force AB at joint 1 is downward, *i.e.* in the direction *ab* in the vector diagram corresponding to a contra-clockwise order, A to B, round joint 1. This is, then, the characteristic order (contra-clockwise) for all the joints, *e.g.* to find the nature of the stress in KL, the order of letters for joint 5 is K to L (contra-clockwise), and referring to the vector

diagram, the direction *k* to *l* represents a thrust of the bar KL on joint 5; the bar KL is therefore in compression.

165. Simple Roof-frame. — Fig. 190 shows a simple roof-frame and its stress diagram when carrying three equal vertical loads on three joints and supported at the extremities of the span.

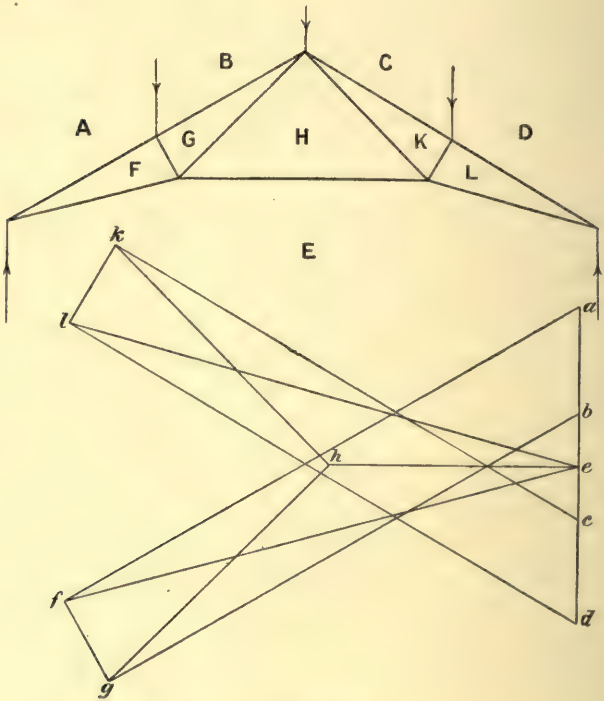


FIG. 190.

The reactions DE and EA at the supports are each obviously equal to half the total load, *i.e.* *e* falls midway between *a* and *d* in the stress diagram. The correct characteristic order of the letters round the joints (Art. 163) is, with the lettering here adopted, clockwise.

166. Loaded Strings and Chains. — Although not coming within the general meaning of the word "frame," stress

diagrams can be drawn for a structure consisting partly of perfectly flexible chains or ropes, provided the loads are such as will cause only tension in flexible members.

Consider a flexible cord or chain, X123Y (Fig. 191), suspended from points X and Y, and having vertical loads of

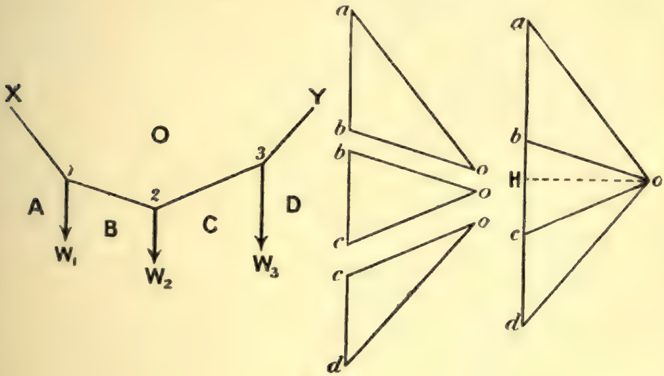


FIG. 191.

W_1, W_2, and W_3 suspended from points 1, 2, and 3 respectively. Denoting the spaces according to Bow's notation by the letters A, B, C, D, and O, as shown above, the tensions in the strings X1 or AO and 1 2 or BO must have a resultant at 1 equal to W_1 vertically upward, to balance the load at 1. If triangles of forces, *abo*, *bco*, and *cdo*, be drawn for the points 1, 2, and 3 respectively, the sides *bo* and *co* appear in two of them, and, as in Art. 163, the three vector triangles may be included in a single vector diagram, as shown at the right-hand by the figure *abcdo*.

The lines *ao*, *bo*, *co*, and *do* represent the tensions in the string crossing the spaces A, B, C, and D respectively. If a horizontal line, *o*H, be drawn from *o* to meet the line *abcd* in H, the length of this line represents the horizontal component of the tensions in the strings, which is evidently constant throughout the whole. (The tension changes only from one space to the neighbouring one by the vector addition of the intermediate vertical load.) The pull on the support X is represented by

ao, the vertical component of which is aH; the pull on Y is represented by od, the vertical component of which is Hd.

A comparison with Art. 157 will show that the various sections of the string X123Y are in the same lines as the sides of a funicular polygon for the vertical forces W_1, W_2, and W_3, corresponding to the pole o. If different lengths of string are attached to X and Y and carry the same loads, W_1, W_2, and W_3, in the lines AB, BC, and CD respectively, they will have different configurations; the longer the string the steeper will be its various slopes corresponding to shorter pole distances, Ho, *i.e.* to smaller horizontal tensions throughout. A short string will involve a great distance of the pole o from the line $abcd$, *i.e.* a great horizontal tension, with smaller inclinations of the various sections of the string. The reader should sketch for himself the shape of a string connecting X to Y, with various values of the horizontal tension Ho, the vertical loads remaining unaltered, in order to appreciate fully how great are the tensions in a very short string.

A chain with hinged links, carrying vertical loads at the joints, will occupy the same shape as a string of the same length carrying the same loads. Such chains are used in suspension bridges.

The shape of the string or chain to carry given loads in assigned vertical lines of action can readily be found for any given horizontal tension, Ho, by drawing the various sections parallel to the corresponding lines radiating from o, *e.g.* AO or X1 parallel to ao (Fig. 191).

Example 1.—A string hangs from two points, X and Y, 5 feet apart, X being 3 feet above Y. Loads of 5, 3, and 4 lbs. are attached to the string so that their lines of action are 1, 2, and 3 feet respectively from X. If the horizontal tension of the string is 6 lbs., draw its shape.

The horizontal distance ZY (Fig. 192) of X from Y is—

$$\sqrt{5^2 - 3^2} = 4 \text{ feet}$$

so that the three loads divide the horizontal span into four equal parts.

Let V_X and V_Y be the vertical components of the tension of the string at X and Y respectively.

The horizontal tension is constant, and equal to 6 lbs. Taking moments about Y (Fig. 192)—

Clockwise. Contra-clockwise.

$$V_X \times 4 = (4 \times 1) + (3 \times 2) + (5 \times 3) + (6 \times 3) \text{ lb.-feet}$$
$$4V_X = 4 + 6 + 15 + 18 = 43$$
$$V_X = \tfrac{43}{4} = 10{\cdot}75 \text{ lbs.}$$

Since the vertical and horizontal components of the tension of the string at X are known, its direction is known. The direction of each section of string might similarly be found. Set out the

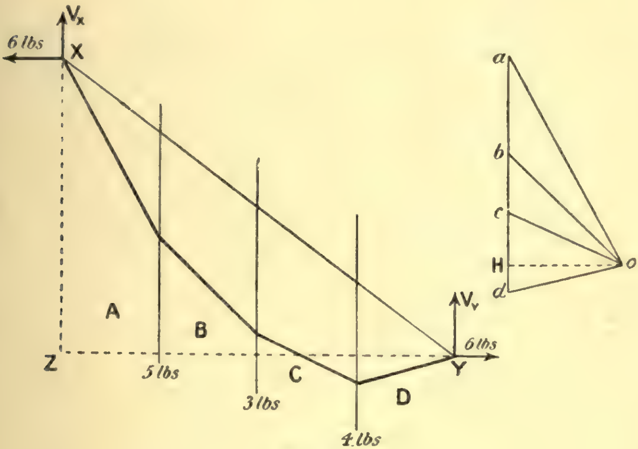


FIG. 192.

vector polygon *abcd*, and draw the horizontal line H*o* to represent 6 lbs. horizontal tension from H, *a*H being measured along *abcd* of such a length as to represent the vertical component 10·75 lbs. of the string at X. Join *o* to *a*, *b*, *c*, and *d*. Starting from X or Y, draw in the lines across spaces A, B, C, and D parallel respectively to *ao*, *bo*, *co*, and *do* (as in Art. 157). The funicular polygon so drawn is the shape of the string.

Example 2.—A chain is attached to two points, X and Y, X being 1 foot above Y and 7 feet horizontally from it. Weights of 20, 27, and 22 lbs. are to be hung on the chain at horizontal distances of 2, 4, and 6 feet from X. The chain is to pass through a point P in the vertical plane of X and Y, 4 feet below, and 3 feet

horizontally from X. Find the shape of the chain and the tensions at its ends.

Let V_X and V_Y be the vertical components of the tension at X and Y respectively, and let H be the constant horizontal tension throughout.

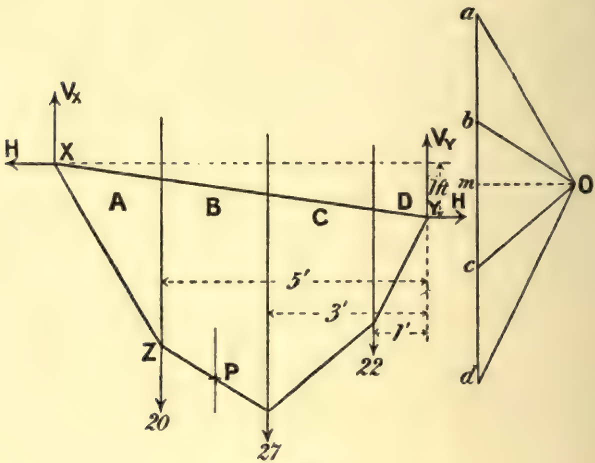


FIG. 193.

Taking moments about Y (Fig. 193)—

$$\begin{array}{c} \text{Clockwise.} \qquad\qquad\qquad \text{Contra-clockwise.} \\ V_X \times 7 = (H \times 1) + (20 \times 5) + (27 \times 3) + (22 \times 1) \\ 7V_X = H + 203 \text{ lbs.-feet} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{array}$$

Taking moments about P of the forces on the chain between X and P—

$$\begin{array}{rl} & \text{Clockwise.} \qquad \text{Contra-clockwise.} \\ V_X \times 3 &= H \times 4 + (20 \times 1) \\ 3V_X &= 4H + 20 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \\ \text{and } 28V_X &= 4H + 812 \text{ from } (1) \\ \text{hence } 25V_X &= 792 \\ V_X &= 31{\cdot}68 \text{ lbs.} \\ H &= 7V_X - 203 = 221{\cdot}76 - 203 = 18{\cdot}76 \text{ lbs.} \end{array}$$

Draw the open polygon of forces, *abcd* (a straight line), and set

off *am* from *a* to the same scale, 31·68 lbs. downwards. From *m* set off *mo* to represent 18·76 lbs. horizontally to the right of *m*.

Then the vector $ao = am + mo$ = tension in the string XZ, which pulls at X in the direction XZ. By drawing XZ parallel to *ao* the direction of the first section of the chain is obtained, and by drawing from Z a line parallel to *bo* to meet the line of action BC, the second section is outlined. Similarly, by continuing the polygon by lines parallel to *co* and *do* the complete shape of the chain between X and Y is obtained.

The tension *ao* at X scales 37 lbs., and the tension *od* at Y scales 44 lbs.

167. Distributed Load.—If the number of points at which the same total load is attached to the string (Fig. 191) be increased, the funicular polygon corresponding to its shape will have a larger number of shorter sides, approximating, if the number of loads be increased indefinitely, to a smooth curve. This case corresponds to that of a heavy chain or string hanging between two points with no vertical load but its own weight. If the dip of the chain from the straight line joining the points of the attachment is small, the load per unit of horizontal span is nearly uniform provided the weight of chain per unit length is uniform. In this case an approximation to the shape of the chain may be found by dividing the span into a number of sections of equal length and taking the load on each portion as concentrated at the mid-point of that section. The funicular polygon for such a system of loads will have one side more than the number into which the span has been divided; the approximation may be made closer by taking more parts. The true curve has all the sides of all such polygons as tangents, or is the curve inscribed in such a polygon.

The polygons obtained by dividing a span into one, two, and four equal parts, and the approximate true curve for a uniform string stretched with a moderate tension, are shown in Fig. 194.

Note that the dip QP would be less if the tensions OH, OA, etc., were increased.

168. The relations between the dip, weight, and tension of a stretched string or chain, assuming perfect flexibility, can

more conveniently be found by ordinary calculation than by graphical methods.

Assuming that the dip is small and the load per horizontal

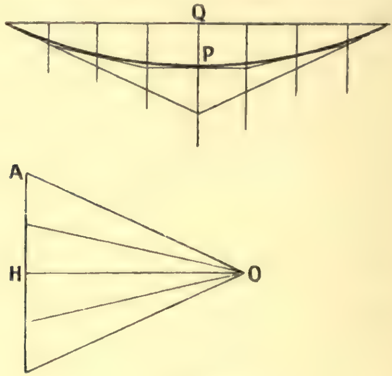


FIG. 194.

foot of span is uniform throughout, the equilibrium of a portion AP (Fig. 195) of horizontal length x, measured from the lowest point A, may be considered.

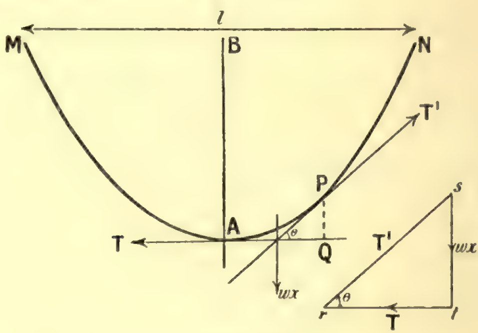


FIG. 195.

Let w = weight per unit horizontal length of cord or chain ;
y = vertical height of P above A, viz. PQ (Fig. 195) ;
T = the tension (which is horizontal) at A ;
T′ = the tension at P acting in a line tangential to the curve at P.

The weight of portion AP is then wx, and the line of action of the resultant weight is midway between AB and PQ, *i.e.* at a distance $\frac{x}{2}$ from either.

Taking moments about the point P—

$$T \times PQ = wx \times \frac{x}{2}$$

$$\text{or } T \times y = \frac{wx^2}{2}$$

$$y = \frac{wx^2}{2T}$$

This relation shows that the curve of the string is a parabola.

If $d =$ the total dip AB, and $l =$ the span of the string or chain, taking moments about N of the forces on the portion AN—

$$d = \frac{w\left(\frac{l}{2}\right)^2}{2T} = \frac{wl^2}{8T}, \text{ or } T = \frac{wl^2}{8d}$$

which gives the relation between the dip, the span, and the horizontal tension.

Returning to the portion AP, if the vector triangle rst be drawn for the forces acting upon it, the angle θ which the tangent to the curve at P makes with the horizontal is given by the relation—

$$\frac{xw}{T} = \frac{st}{tr} = \tan\theta$$

Also the tension T′ at P is T sec θ, or—

$$T' = T\sqrt{1 + \tan^2\theta} = T\sqrt{1 + \left(\frac{wx}{T}\right)^2} = \sqrt{T^2 + w^2x^2}$$

and at the ends where $x = \frac{l}{2}$—

$$T' = \sqrt{T^2 + \frac{w^2l^2}{4}}$$

And since $T = \frac{wl^2}{8d}$, the tension at N or M is—

$$\frac{wl}{2}\sqrt{\frac{l^2}{16d^2} + 1}, \text{ or } \frac{wl^2}{8d}\sqrt{1 + \frac{16d^2}{l^2}}$$

which does not greatly exceed $\frac{wl^2}{8d}$ (or T), if $\frac{d}{l}$ is small.

Example.—A copper trolley-wire weighs $\frac{1}{2}$ lb. per foot length; it is stretched between two poles 50 feet apart, and has a horizontal tension of 2000 lbs. Find the dip in the middle of the span.

Let d = the dip in feet.

The weight of the wire in the half-span BC (Fig. 196) is $25 \times \frac{1}{2} = 12{\cdot}5$ lbs.

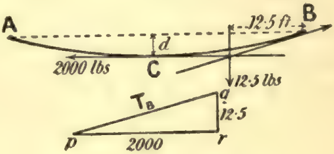


Fig. 196.

The distance of the c.g. of the wire BC from B is practically 12·5 feet horizontally.

Taking moments about B of the forces on the portion BC—

$$2000 \times d = 12{\cdot}5 \times 12{\cdot}5$$
$$d = 0{\cdot}07812 \text{ foot} = 0{\cdot}938 \text{ inch}$$

Examples XX.

1. A roof principal, shown in Fig. 197, carries loads of 4, 7, and 5 tons in the positions shown. It is simply supported at the extremities of a span

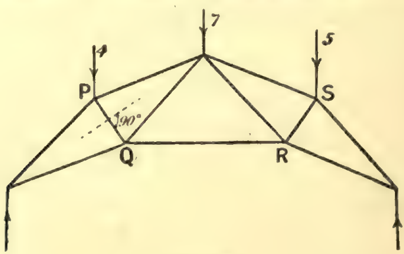


Fig. 197.

of 40 feet. The total rise of the roof is 14 feet, and the distances PQ and

RS are each 5·4 feet. Draw the stress diagram and find the stress in each member of the frame.

2. A Warren girder (Fig. 198), made up of bars of equal lengths, carries a single load of 5 tons as shown. Draw the stress diagram and scale off

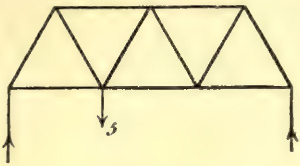


FIG. 198.

the forces in each member; check the results by the method of sections (Art. 98).

3. Draw the stress diagram for the roof-frame in Fig. 199 under the

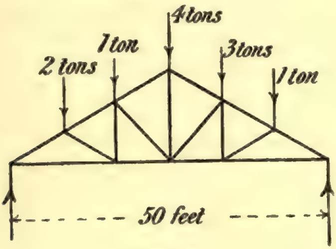


FIG. 199.

given loads. The main rafters are inclined at 30° to the horizontal, and are each divided by the joints into three equal lengths.

4. A chain connects two points on the same level and 10 feet apart; it has suspended from it four loads, each of 50 lbs., at equal horizontal intervals along the span. If the tension in the middle section is 90 lbs., draw the shape of the chain, measure the inclination to the horizontal, and the tension of the end section.

5. Find the shape of a string connecting two points 8 feet horizontally apart, one being 1 foot above the other, when it has suspended from it weights of 5, 7, and 4 lbs. at horizontal distances of 2, 5, and 6 feet respectively from the higher end, the horizontal tension of the string being 6 lbs.

6. A light chain connects two points, X and Y, 12 feet horizontally apart, X being 2 feet above Y. Loads of 15, 20, and 25 lbs. are suspended from the chain at horizontal distances of 3, 5, and 8 feet respectively from X. The chain passes through a point 7 feet horizontally from X and 4 feet

below it. Draw the shape of the chain. How far is the point of suspension of the 15-lb. load from X?

7. A wire is stretched horizontally, with a tension of 50 lbs., between two posts 60 feet apart. If the wire weighs 0·03 lb. per foot, find the sag of the wire in inches.

8. A wire weighing 0·01 lb. per foot is stretched between posts 40 feet apart. What must be the tension in the wire in order to reduce the sag to 2 inches?

9. A wire which must not be stretched with a tension exceeding 70 lbs. is to be carried on supporting poles, and the sag between two poles is not to exceed 1·5 inches. If the weight of the wire is 0·025 lb. per foot, find the greatest distance the poles may be placed apart.

APPENDIX

UNITS AND THEIR DIMENSIONS

Units.—To express the magnitude of any physical quantity it has to be stated in terms of a unit of its own kind. Thus by stating that a stick is 2·75 feet long, we are using the foot as the unit of length.

Fundamental and Derived Units.—We have seen that the different quantities in common use in the science of mechanics have certain relations to one another. If the units of certain selected quantities are arbitrarily fixed, it is possible to determine the units of other quantities by means of their relations to the selected ones. The units arbitrarily fixed are spoken of as fundamental units, and those depending upon them as derived units.

Fundamental Units.—There are two systems of units in general use in this country. In the C.G.S. system (Art. 42), which is commonly used in physical science, the units chosen as fundamental and arbitrarily fixed are those of length, mass, and time, viz. the centimetre, gramme, and second.

In the British gravitational system the fundamental units chosen are those of length, force, and time, viz. the foot, the pound, *i.e.* the weight of 1 lb. of matter at some standard place, and the mean solar second.

The latter system of units has every claim to the name "absolute," for three units are fixed, and the other mechanical units are derived from them by fixed relations.

The *weight* of a body of given mass varies at different parts of the earth's surface in whatever units its mass is measured. The value of 1 lb. force, however, does *not* vary, since it has been defined as the weight of a fixed mass at a *fixed place*.

Dimensions of Derived Units.

(*a*) *Length—Mass—Time Systems.*—In any such system other than, say, the C.G.S. system, let the unit of length be L centimetres, the unit of mass M grammes, and the unit of time be T seconds.

Then the unit of area will be $L \times L$ or L^2 square centimetres, *i.e.* it varies as the square of the magnitude of the unit of length. Similarly, we may derive the other important mechanical units as follows :—

Unit volume $= L \times L \times L$ or L^3 cubic centimetres, or unit volume varies as L^3.

Unit velocity is L centimetres in T seconds $= \frac{L}{T}$ centimetres per second, or LT^{-1} centimetres per second.

Unit acceleration is $\frac{L}{T}$ centimetres per second in T seconds $= \frac{L}{T^2}$ centimetres per second, or LT^{-2} centimetres per second.

Unit momentum is that of M grammes moving $\frac{L}{T}$ centimetres per second, *i.e.* $\frac{ML}{T}$ C.G.S. units of momentum, or MLT^{-1} C.G.S. units.

Unit force is unit change of momentum in T seconds, or $\frac{MLT^{-1}}{T}$ units in one second, or MLT^{-2} dynes (C.G.S. units of force).

Unit impulse is given by unit force (MLT^{-2} C.G.S. units) acting for unit time, T seconds generating a change of momentum (or impulse) MLT^{-1} C.G.S. units.

Unit work is that done by unit force (MLT^{-2} dynes) acting through L centimetres, *i.e.* ML^2T^{-2} centimetre-dynes or ergs.

Unit kinetic energy is that possessed by unit mass, M grammes moving with unit velocity (LT^{-1}), *i.e.* $\frac{1}{2}M(LT^{-1})^2 = \frac{1}{2}ML^2T^{-2}$ C.G.S. units.

Unit power is unit work in unit time T seconds, or $\frac{ML^2T^{-2}}{T}$, or ML^2T^{-3} ergs per second.

Note that unit angle $\frac{\text{L units of arc}}{\text{L units radius}} = 1$ radian, and is independent of the units of length, mass, or time.

Unit angular velocity is unit velocity $\frac{L}{T}$ divided by unit radius L centimetres, or $LT^{-1} \div L = T^{-1}$.

Unit moment of momentum or angular momentum is unit momentum MLT^{-1} at unit perpendicular distance L, or ML^2T^{-1} C.G.S. units.

Unit moment of force is unit force MLT^{-2} at unit distance L centimetres, or ML^2T^{-2} C.G.S. units.

Unit rate of change of angular momentum is ML^2T^{-1} C.G.S. units in unit time T seconds = ML^2T^{-2} C.G.S. units.

Unit moment of inertia is that of unit mass M grammes at unit distance L centimetres, which is ML^2 C.G.S. units.

Thus each derived unit depends on certain powers of the magnitudes of the fundamental units, or has certain dimensions of those units.

(*b*) The dimensions of the same quantities in terms of the three fundamental units of length, force, and time may be similarly written as follows :—

	Quantity.	Dimensions.
Fundamental.	Length.	L.
	Force.	F.
	Time.	T.
	Velocity.	$\frac{L}{T}$ or LT^{-1}.
	Acceleration.	$\frac{L}{T} \div T$ or LT^{-2}.
	Mass.	$\frac{\text{Force}}{\text{Acceleration}}$ or $FL^{-1}T^2$.
	Momentum.	FT.
	Impulse.	FT.
	Work.	FL.
	Kinetic energy.	$\frac{1}{2}$FL.
	Power.	FLT^{-1}.
	Angular momentum.	FLT.
	Moment of force.	FL.
	Rate of change of angular momentum.	FL.

Symbolical formulæ and equations may be checked by testing if the dimensions of the terms are correct. Each term on either side of an algebraic equation having a physical meaning must necessarily be of the same dimensions.

ANSWERS TO EXAMPLES

Examples I.

(1) 0·30$\dot{5}$ foot per second per second. (2) 5·5 seconds; 121 feet.
(3) 71·77 feet per second. (4) 3·053 seconds.
(5) 89·5 feet; 447·5 feet; 440·4 feet.
(6) 5·63 seconds after the first projection; 278 feet.
(7) 56·7 feet per second. (8) 4·5, 14·6, and 11·4 feet per second.
(10) 0·57 and 0·393 foot per second per second; 880 feet.
(11) 77·3 feet; 2·9 seconds.

Examples II.

(1) 4·88 feet per second; 35° 23′ to the horizontal velocity.
(2) 405 feet per second; 294 feet per second.
(3) 53° up-stream; 2 minutes 16·4 seconds. (4) 10° 6 south of west.
(5) 19·54 knots per hour; 5 hours 7·2 minutes; 12° 8′ west of south.
(6) 48 minutes; 9·6 miles; 12·8 miles.
(7) 154·2 feet per second per second; 21°·5 south of west.
(8) 2·59 seconds. (9) 5·04; 4·716. (10) 16·83 feet per second.
(11) 35·2 radians per second; 2·581 radians per second per second.
(12) 135 revolutions and 1·5 minutes from full speed.

Examples III.

(1) 2735 units; 182,333 lbs. or 81·4 tons. (2) $\frac{75}{64}$ or 1·172 to 1.
(3) 2·8 centimetres per second. (4) 9802 lbs.
(5) 15·33 lbs.; 9·53 units per second in direction of jet; 9·53 lbs.
(6) 45·3. (7) 4720 lbs.
(8) 10·43 tons inclined downwards at 16° 40′ to horizontal.
(9) 2·91 units; 727·5 lbs. (10) 8750 units; 8·57 miles per hour.

Examples IV.

(1) 67·8 lbs. (2) 17·48 lbs. (4) 34·54 feet.
(5) 23·44 feet per second; 255,000 lbs. (6) 1005 feet per second.
(7) 154 lbs.; 126 lbs.; 6·9 feet per second per second.
(8) 11·243 cwt. (9) 9·66 feet; 14·9$\dot{3}$ lbs.
(10) 4·69 grammes; 477 centimetres.
(11) 6·44 feet per second per second; 4 lbs.
(12) 1·027 lbs. (13) 48·9 lbs.

EXAMPLES V.

(1) 160 horse-power ; 303·36 horse-power ; 16·64 horse-power.
(2) 15·75 lbs. per ton.
(3) 22·15 miles per hour.
(4) 929 ; 1253.
(5) 147·5 horse-power.
(6) 0·347 horse-power.
(7) 60 foot-lbs.
(8) 350,000 foot-lbs. ; 800,000 foot-lbs.
(9) 1,360,000 foot-lbs.

EXAMPLES VI.

(1) 57·1 horse-power.
(2) 39,390 lb.-feet.
(3) 6570 lb.-feet.
(4) 609 inch-lbs.
(5) 5340 inch-lbs. ; 2220 inch-lbs.
(6) 12·8 horse-power.

EXAMPLES VII.

(1) 12,420,000 foot-lbs. ; 4,140,000 lbs.
(2) 27·8 feet per second.
(3) 37,740 inch-lbs. ; 35,940 inch-lbs.
(4) 25·5 horse-power.
(5) 7·02 horse-power.
(6) 7·25 horse-power.
(7) 19·6 horse-power.
(8) 8·65 seconds.
(9) 10·5 feet per second ; 467 lbs.
(10) 15·3 seconds.
(11) 2886 foot-lbs.
(12) 500,000 foot-lbs.

EXAMPLES VIII.

(1) 68·5
(2) 11·85 miles per hour.
(3) 2672 feet.
(4) 4·25 inches.
(5) 3052 feet.
(6) 20 miles per hour.
(7) 47° to horizontal.
(8) $52^\circ{\cdot}5$; 1·64 times the weight of the stone.
(9) 1·5 per cent. increase.
(10) 66·4 ; 72·7 ; 59·3 revolutions per minute.
(11) $6^\circ{\cdot}1$.
(12) 38·33 ; 35·68 feet per second, 7·79 ; 6·28 lbs.

EXAMPLES IX.

(1) 0 855, 1·56, 1·81 feet per second ; 8·05, 5·96, 4·4 feet per second per second.
(2) $\frac{3}{4}$ inch.
(3) 1654, 827, 1474 lbs.
(4) 153·3.
(6) 0·342 second.
(7) 1·103 second ; 67·3 feet per second per second.
(8) 31·23.
(9) 1 to 1·0073.

EXAMPLES X.

(1) 14·65 lbs. ; 17·9 lbs.
(2) 3 lbs. ; 13 lbs.
(3) 9·6 tons tension ; 55·6 tons tension.
(4) $41^\circ{\cdot}7$ south of west ; 720 lbs.
(5) 2250 lbs. ; 2890 lbs.
(6) 220 lbs. ; 58·5 lbs.

Examples XI.

(1) 0·15̇4 ; 8°·8 (2) 2·97 lbs. ; 8°·5 to horizontal. (3) 14·51 lbs.
(4) 0·6 times the weight of log ; 36°·8 to horizontal. (5) 10°·4
(6) 0·3066 horse-power. (7) 179 horse-power.
(8) 3·84 horse-power.
(9) 3·4 feet per second per second ; 3·57 lbs.
(10) 4·5 tons ; 31·9 seconds. (11) 3820 lbs.

Examples XII.

(1) 261 lbs. (2) 16·97 lbs. ; 4·12 lbs.
(3) Left, 5·242 tons ; right, 5·008 tons.
(4) Left, 10 tons ; right, 3 tons ; end, 2·824 tons.
(5) 1·039 inches. (6) 5·737 feet from end.

Examples XIII.

(1) Tension, 21·68 lbs. ; pressure, 33·4 lbs. ; 19°·7 to vertical
(2) 0·1264. (3) 36°.
(4) 15·3 lbs. at hinge ; 8·25 lbs. at free end.
(5) 3950 lbs. at A ; 2954 lbs. at C.
(6) 11·2 lbs. cutting AD 2·1 inches from A, inclined 19°·3 to DA.
(7) 4·3 tons ; 3·46 tons ; 46·7° to horizontal.
(8) 8·2 tons compression ; 4·39 tons tension ; 4 tons tension.
(9) 8·78 tons tension ; 25·6 tons compression ; 21·22 tons tension.

Examples XIV.

(1) 1·27 feet from middle. (2) 2·08 inches.
(3) 43 inches. (4) 1·633 feet ; 1·225 feet.
(5) 4·18 inches ; 4·08 inches. (6) 10·1 inches ; 5·5 lbs.
(7) 2·98 inches. (8) 27·2 inches.
(9) 9·75 inches. (10) 1293 lb.-feet ; 103·5 lbs. per square foot.
(11) 11·91 inches. (12) 4·82 inches.
(13) 4 feet 5·1 inches. (14) 0·17 lb. (15) 0·197 lb. ; 0·384 lb.

Examples XV.

(1) 19·48 inches ; 16·98 inches. (2) 12·16 inches.
(3) 6·08 inches. (4) 15·4 inches.
(5) 2·52 inches from outside of flange. (6) 4·76 inches.
(7) 0·202 inch from centre. (8) 16·6 inches.
(9) 5·36 inches. (10) 33·99 inches.

EXAMPLES XVI

(1) 16 and 8 tons.
(2) 25 and 16 tons.
(3) Left, 16·5 tons; right, 33·4 tons.
(4) 53° 10′.
(5) 16·43 inches; 4·41 inches.
(6) 3·53 inches.
(7) 3·67 inches.
(8) 8000 foot-lbs.
(9) 1188 foot-lbs.
(10) 140,000; 74,400 foot-lbs.
(11) 75,600 foot-lbs.
(12) 2514 foot-lbs.
(13) 110·3 lbs.
(14) 5·11 lbs.
(15) 37·6 square inches.
(16) 7·85 cubic inches.
(17) 4 feet 3·9 inches.

EXAMPLES XVII.

(1) 312 $(\text{inches})^4$.
(2) 405 $(\text{inches})^4$; 4·29 inches.
(3) 195 $(\text{inches})^4$; 2·98 inches.
(4) 290 $(\text{inches})^4$.
(5) 5·523 inches.
(6) 0·887 gravitational units.
(7) $\sqrt{\frac{a^2 + b^2}{2}}$.
(8) 16·1 inches; 35·15 gravitational units.

EXAMPLES XVIII.

(1) 3647 gravitational units.
(2) 13,215 gravitational units.
(3) 10 minutes 46 seconds; 323.
(4) 17·48 lbs.
(5) 350 lb.-feet.
(6) 2·134 gravitational units; 6·83 inches.
(7) 141·3.
(8) 7·71 inches.
(9) 22 feet per second; 31·06 feet per second.
(10) 14·85 feet per second; 16·94 feet per second.
(11) 3·314 feet; 3819 gravitational units.
(12) 53·7.
(13) 0·0274 units.
(14) 125·5.
(15) 117·5 foot-tons.
(16) 16·7 feet per second.
(17) 23 sin α feet per second per second.
(18) $\left(W + 4w \cdot \frac{k^2}{a^2}\right)\frac{v^2}{2g}$.

EXAMPLES XIX.

(1) 6·47 lbs.; 0·016 inch; 102°·6.
(2) 7·8 lbs.; 0·013 inch; 36°.
(3) 17·7 right; 11·3 left.
(4) 21·6 lbs.; 134° measured clockwise.
(5) 30·4, 38·15, 34·85 tons-feet.
(6) 38·15, 34·85, 29·6, 25·95 tons-feet.
(7) 1·65 tons; 2·35 tons; 3·65 tons.
(8) 25 tons-feet; nil; 18·75 tons-feet; 2·5 tons.

EXAMPLES XX.

(4) 48°; 134·5 lbs.
(6) 4·06 feet.
(7) 3·24 inches.
(8) 12 lbs.
(9) 53 feet.

EXAMINATION QUESTIONS

Questions selected from the Mechanics Examinations Intermediate (Engineering) Science of London University.

1. What is implied in the rule: the product of the diameter of a wheel in feet, and of the revolutions per minute, divided by 28, is the speed in miles an hour?

Also in the rule: three times the number of telegraph posts per minute is the speed in miles an hour? (1903.)

2. Continuous breaks are now capable of reducing the speed of a train $3\frac{3}{4}$ miles an hour every second, and take 2 seconds to be applied. Show in a tabular form the length of an emergency stop at speeds of $3\frac{3}{4}$, $7\frac{1}{2}$, 15, 30, 45, and 60 miles an hour.

Compare the retardation with gravity; express the resisting force in pounds per ton; calculate the coefficient of adhesion of the break shoe and rail with the wheel; and sketch the arrangement. (1903.)

3. Prove that the horse-power required to overcome a resistance of R lbs. at a speed of S miles an hour is RS ÷ 375. Calculate the horse-power of a locomotive drawing a train of 200 tons up an incline of 1 in 200 at 50 miles an hour, taking the road and air resistance at this speed at 28 lbs. a ton. (1903.)

4. If W tons is transported from rest to rest a distance s feet in t seconds, being accelerated for a distance s_1 and time t_1 by a force P_1 tons up to velocity v feet per second, and then brought to rest by a force P_2 tons acting for t_2 seconds through s_2 feet—

$$\text{(i.)}\quad \frac{Wv}{g} = P_1t_1 = P_2t_2 = \frac{P_1P_2}{P_1 + P_2}t$$

$$\text{(ii.)}\quad \frac{Wv^2}{2g} = P_1s_1 = P_2s_2 = \frac{P_1P_2}{P_1 + P_2}s$$

$$\text{(iii.)}\quad v = 2\frac{s_1}{t_1} = 2\frac{s_2}{t_2} = 2\frac{s}{t}$$

A train of 100 tons gross, fitted with continuous breaks, is to be run on a level line between stations one-third of a mile apart, at an average speed of 12 miles an hour, including two-thirds of a minute stop at each station. Prove that the weight on the driving wheels must exceed $22\frac{1}{2}$ tons, with an adhesion of one-sixth, neglecting road-resistance and delay in application of the breaks. (1903.)

5. Give a graphical representation of the relative motion of a piston and crank, when the connecting rod is long enough for its obliquity to be neglected; and prove that at R revolutions a minute, the piston velocity is $\frac{\pi R}{30}$ times the geometric mean of the distance from the two ends of the stroke.

Prove that if the piston weighs W lbs., the force in pounds which gives its acceleration is—

$$\frac{W}{g} \cdot \frac{\pi^2 R^2}{900} \text{ (distance in feet from mid-point of stroke).}$$

(1903.)

6. Write down the formula for the time of swing of a simple pendulum, and calculate the percentage of its change due to 1 per cent. change in length or gravity, or both.

Prove that the line in Question 4 could be worked principally by gravity if the road is curved downward between the stations to a radius of about 11,740 feet, implying a dip of 33 feet between the stations, a gradient at the stations of 1 in 13, and a maximum running velocity of 31 miles an hour. (1903.)

7. Prove that if a hammer weighing W tons falling h feet drives a pile weighing w tons a feet into the ground, the average resistance of the ground in tons is—

$$\frac{W^2}{W + w} \cdot \frac{h}{a}$$

Prove that the energy dissipated at the impact is diminished by increasing—

$$\frac{W}{w} \qquad \text{(1903.)}$$

8. Prove that the total kinetic energy stored up in a train of railway carriages, weighing W tons gross, when moving at v feet per second is—

$$\left(W + W_1\frac{k^2}{a^2}\right)\frac{v^2}{2g} \text{ foot-tons}$$

where W_1 denotes the weight of the wheels in tons, a their radius, and k their radius of gyration.

Prove that W in the equations of Question 4 must be increased by $\frac{W_1 k^2}{a^2}$ to allow for the rotary inertia of the wheels. (1903.)

9. Determine graphically, by the funicular polygon, the reaction of the supports of a horizontal beam, loaded with given weights at two given points.

Prove that the bending moment at any point of the beam is represented by the vertical depth of the funicular polygon.

(1903.)

10. A wheel is making 200 revolutions per minute, and after 10 seconds its speed has fallen to 150 revolutions per minute. If the angular retardation be constant, how many more revolutions will it make before coming to rest? (1904.)

11. A piston is connected to a flywheel by a crank and connecting rod in the usual manner. If the angular velocity of the flywheel be constant, show that if the connecting rod be sufficiently long the motion of the piston will be approximately simple-harmonic; and find the velocity of the piston in any position.

If A, B, C, D, E be five equidistant positions of the piston, A and E being the ends of its stroke, prove that the piston takes twice as long to move from A to B as it does from B to C.

(1904.)

12. A train whose weight is 250 tons runs at a uniform speed down an incline of 1 in 200, the steam being shut off and the brakes not applied, and on reaching the foot of the incline it runs 800 yards on the level before coming to rest. What was its original speed in miles per hour?

[The frictional resistance is supposed to be the same in each case.] (1904.)

13. A weight A hangs by a string and makes small lateral oscillations like a pendulum; another weight, B, is suspended by a spiral spring, and makes vertical oscillations. Explain why an addition to B alters the period of its oscillations, whilst an addition to A does not. Also find exactly how the period of B varies with the weight. (1904.)

14. A steel disc of thickness t and outer radius a is keyed on to a cylindrical steel shaft of radius b and length l, and the centre of the disc is at a distance c from one end of the shaft. Find the distance from this end of the shaft of the mass-centre of the whole.

(1904.)

15. A uniform bar 6 feet long can turn freely in a vertical plane about a horizontal axis through one end. If it be just started from

the position of unstable equilibrium, find (in feet per second) the velocity of the free end at the instant of passing through its lowest position. (1904.)

16. Explain why, as a man ascends a ladder, the tendency of its foot to slip increases.

A man weighing 13 stone stands on the top of a ladder 20 feet long, its foot being 6 feet from the wall. How much is the horizontal pressure of the foot on the ground increased by his presence, the pressure on the wall being assumed to be horizontal? (1904.)

17. A horizontal beam 20 feet long, supported at the ends, carries loads of 3, 2, 5, 4 cwts. at distances of 3, 7, 12, 15 feet respectively from one end. Find by means of a funicular polygon (drawn to scale) the pressures on the two ends, and test the accuracy of your drawing by numerical computation. (1904.)

Questions selected from the Associate Members' Examinations of the Institution of Civil Engineers.

1. A beam 20 feet long is supported on two supports 3 feet from each end of the beam; weights of 10 lbs. and 20 lbs. are suspended from the two ends of the beam. Draw, to scale, the bending-moment and shearing-force diagrams; and, in particular, estimate their values at the central section of the beam. (I.C.E., February, 1905.)

2. The speed of a motor car is determined by observing the times of passing a number of marks placed 500 feet apart. The time of traversing the distance between the first and second posts was 20 seconds, and between the second and third 19 seconds. If the acceleration of the car is constant, find its magnitude in feet per second per second, and also the velocity in miles per hour at the instant it passes the first post. (I.C.E., February, 1905.)

3. In a bicycle, the length of the cranks is 7 inches, the diameter of the back wheel is 28 inches, and the gearing is such that the wheel rotates $2\frac{1}{2}$ times as fast as the pedals. If the weight of the cyclist and machine together is 160 lbs., estimate the force which will have to be applied to the pedal to increase the speed uniformly from 4 to 12 miles an hour in 20 seconds, frictional losses being neglected. (I.C.E., February, 1905.)

4. A thin circular disc, 12 inches radius, has a projecting axle $\frac{1}{2}$ inch diameter on either side. The ends of this axle rest on two parallel inclined straight edges inclined at a slope of 1 in 40, the

lower part of the disc hanging between the two. The disc rolls from rest through 1 foot in $53\frac{1}{2}$ seconds. Neglecting the weight of the axle and frictional resistances, find the value of g.

(I.C.E., February, 1905.)

5. A gate 6 feet high and 4 feet wide, weighing 100 lbs., hangs from a rail by 2 wheels at its upper corners. The left-hand wheel having seized, skids along the rail with a coefficient of friction of $\frac{1}{3}$. The other wheel is frictionless. Find the horizontal force that will push the gate steadily along from left to right if applied 2 feet below the rail. You may solve either analytically or by the force-and-link polygon. (I.C.E., October, 1904.)

6. A ladder, whose centre of gravity is at the middle of its length, rests on the ground and against a vertical wall; the coefficients of friction of the ladder against both being $\frac{1}{4}$. Find the ladder's inclination to the ground when just on the point of slipping.

(I.C.E., October, 1904.)

7. The faceplate of a lathe has a rectangular slab of cast iron bolted to it, and rotates at 480 revolutions per minute. The slab is 8 inches by 12 inches by 30 inches (the length being radial). Its outside is flush with the edge of the faceplate, which is 48 inches diameter. Find the centrifugal force. (Cast iron weighs $\frac{1}{4}$ lb. per cubic inch.) Where must a circular weight of 300 lbs. be placed to balance the slab? (I.C.E., October, 1904.)

8. A boom 30 feet long, weighing 2 tons, is hinged at one end, and is being lowered by a rope at the other. When just horizontal the rope snaps. Find the reaction on the hinge.

(I.C.E., October, 1904.)

9. A beam ABCD, whose length, AD, is 50 feet, is supported at each end, and carries a weight of 2 tons at B, 10 feet from A, and a weight of 2·5 tons at C, 20 feet from D. Calculate the shearing force at the centre of the span, and sketch the diagram of shearing forces. (I.C.E., October, 1904.)

10. Referring to the loaded beam described in the last question, how much additional load would have to be put on at the point B in order to reduce the shearing force at the centre of the span to zero? (I.C.E., October, 1904.)

11. Estimate the super-elevation which ought to be given to the outer rail when a train moves round a curve of 2000 feet radius at a speed of 60 miles an hour, the gauge being 4 feet $8\frac{1}{2}$ inches.

(I.C.E., February, 1904.)

12. Show that in simple-harmonic motion the acceleration is proportional to the displacement from the mid-point of the path, and that the time of a small oscillation of a simple pendulum of

length l is $2\pi\sqrt{\frac{l}{g}}$. Deduce the length of the simple pendulum which has the same time of oscillation as a uniform rod of length L suspended at one end. (I.C.E., February, 1904.)

13. To a passenger in a train moving at the rate of 40 miles an hour, the rain appears to be rushing downwards and towards him at an angle of 20° with the horizontal. If the rain is actually falling in a vertical direction, find the velocity of the rain-drops in feet per second. (I.C.E., February, 1904.)

14. If it take 600 useful horse-power to draw a train of 335 tons up a gradient of 1 in 264 at a uniform speed of 40 miles an hour, estimate the resistance per ton other than that due to ascending against gravity, and deduce the uniform speed on the level when developing the above power. (I.C.E., February, 1904.)

15. In a steam-hammer the diameter of the piston is 36 inches, the total weight of the hammer and piston is 20 tons, and the effective steam pressure is 40 lbs. per square inch. Find the acceleration with which the hammer descends, and its velocity after descending through a distance of 4 feet. If the hammer then come in contact with the iron, and compress it through a distance of 1 inch, find the mean force of compression.

(I.C.E., February, 1904.)

16. A uniform circular plate, 1 foot in diameter and weighing 4 lbs., is hung in a horizontal plane by three fine parallel cords from the ceiling, and when set in small torsional oscillations about a vertical axis is found to have a period of 3 seconds. A body whose moment of inertia is required is laid diametrically across it, and the period is found to be 5 seconds, the weight being 6 lbs. Find the moment of inertia of the body about the axis of oscillation.

(I.C.E., February, 1904.)

17. The acceleration of a train running on the level is found by hanging a short pendulum from the roof of a carriage and noticing the angle which the pendulum makes with the vertical. In one experiment the angle of inclination was 5°: estimate the acceleration of the train in feet per second per second and in miles per hour per hour. (I.C.E., February, 1904.)

18. Two weights, one of 2 lbs. and the other of 1 lb., are connected by a massless string which passes over a smooth peg. Find the tension in the string and the distance moved through by either weight, from rest, in 2 seconds. (I.C.E., February, 1904.)

19. A solid circular cast-iron disc, 20 inches in diameter and 2 inches thick (weighing 0·25 lb. per cubic inch), is mounted on ball bearings. A weight of 10 lbs. is suspended by means of a

string wound round the axle, which is 3 inches in diameter, and the weight is released and disconnected after falling 10 feet. Neglecting friction, find the kinetic energy stored in the wheel, and the revolutions per minute the wheel is making when the weight is disconnected, and also the time it would continue to run against a tangential resistance of $\frac{1}{2}$ lb. applied at the circumference of the axle. (I.C.E., February, 1904.)

20. A girder 20 feet long carries a distributed load of 1 ton per lineal foot over 6 feet of its length, the load commencing at 3 feet from the left-hand abutment. Sketch the shearing-force and bending-moment diagrams, and find, independently, the magnitude of the maximum bending moment and the section at which it occurs. (I.C.E., February, 1904.)

21. Explain what is meant by centripetal acceleration, and find its value when a particle describes a circle of radius r feet with a velocity of v feet per second. (I.C.E., October, 1903.)

22. In an electric railway the average distance between the stations is $\frac{1}{2}$ mile, the running time from start to stop $1\frac{1}{2}$ minutes, and the constant speed between the end of acceleration and beginning of retardation 25 miles an hour. If the acceleration and retardation be taken as uniform and numerically equal, find their values; and if the weight of the train be 150 tons and the frictional resistance 11 lbs. per ton, find the tractive force necessary to start on the level. (I.C.E., October, 1903.)

23. The mass of a flywheel may be assumed concentrated in the rim. If the diameter is 7 feet and the weight $2\frac{1}{2}$ tons, estimate its kinetic energy when running at 250 revolutions per minute. Moreover, if the shaft be 6 inches in diameter and the coefficient of friction of the shaft in the bearings be 0·09, estimate the number of revolutions the flywheel will make before coming to rest.

(I.C.E., October, 1903.)

24. A plane inclined at 20° to the horizontal carries a load of 1000 lbs., and the angle of friction betweeen the load and the plane is 10°. Obtain the least force in magnitude and direction which is necessary to pull the load up the plane.

(I.C.E., October, 1903.)

25. State the second law of motion. A cage weighing 1000 lbs. is being lowered down a mine by a cable. Find the tension in the cable, (1) when the speed is increasing at the rate of 5 feet per second per second; (2) when the speed is uniform; (3) when the speed is diminishing at the rate of 5 feet per second per second. The weight of the cable itself may be neglected.

(I.C.E., October, 1903.)

26. Show that when a helical spring vibrates freely under the action of a weight, its periodic time is the same as that of a simple pendulum having a length equal to the static extension of the spring when carrying the weight, the mass of the spring itself being neglected. (I.C.E., October, 1903.)

27. A flywheel, supported on an axle 2 inches in diameter, is pulled round by a cord wound round the axle and carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome friction. A further weight of 16 lbs., making 20 lbs. in all, is applied, and two seconds after starting from rest it is found that the weight has descended a distance of 4 feet. Estimate the moment of inertia of the wheel about the axis of rotation in gravitational units. (I.C.E., October, 1903.)

28. With an automatic vacuum brake a train weighing 170 tons and going at 60 miles an hour on a down gradient of 1 in 100 was pulled up in a distance of 596 yards. Estimate the total resistance in pounds per ton; and if the retardation is uniform, find the time taken to bring the train to rest. (I.C.E., October, 1903.)

29. A string, ABCD, hangs in a vertical plane, the ends A and D being fixed. A weight of 10 lbs. is hung from the point B, and an unknown weight from the point C. The middle-portion BC is horizontal, and the portions AB and CD are inclined at 30° and 45° to the horizontal respectively. Determine the unknown weight and the tensions in the three portions of the string. (I.C.E, October, 1903.)

30. Two masses, of 10 lbs. and 20 lbs. respectively, are attached to a balanced disc at an angular distance apart of 90° and at radii 2 feet and 3 feet respectively. Find the resultant force on the axis when the disc is making 200 turns a minute; and determine the angular position and magnitude of a mass placed at 2·5 feet radius which will make the force on the axis zero at all speeds. (I.C.E., October, 1903.)

31. A crane has a vertical crane-post, AB, 8 feet long, and a horizontal tie, BC, 6 feet long, AC being the jib. It turns in bearings at A and B, and the chain supporting the load passes over pulleys at C and A, and is then led away at 30° to AB. Find the stresses in the bars and thrusts in the bearings when lifting 1 ton at a uniform rate. (I.C.E., February, 1903.)

32. A man ascends a ladder resting on a rough horizontal floor against a smooth vertical wall. Determine, graphically or otherwise, the direction of the action between the foot of the ladder and the floor. (I.C.E., February, 1903.)

33. A train on a horizontal line of rails is accelerating the speed

uniformly so that a velocity of 60 miles an hour is being acquired in 176 seconds. A heavy weight is suspended freely from the roof of a carriage by a string. Calculate, or determine graphically, the inclination of the string to the vertical.

(I.C.E., February, 1903.)

34. Explain the meaning of the term "centrifugal force." With what speed must a locomotive be running on level railway lines, forming a curve of 968 feet radius, if it produce a horizontal thrust on the outer rail equal to $\frac{1}{64}$ of its weight?

(I.C.E., February, 1903.)

35. Explain how to determine the relative velocity of two bodies. A is travelling due north at constant speed. When B is due west of A, and at a distance of 21 miles from it, B starts travelling north-east with the same constant speed as A. Determine graphically, or otherwise, the least distance which B attains from A.

(I.C.E., February, 1903.)

36. Two men put a railway-waggon weighing 5 tons into motion by exerting on it a force of 80 lbs. The resistance of the waggon is 10 lbs. per ton, or altogether 50 lbs. How far will the waggon have moved in one minute? Calculate at what fraction of a horse-power the men are working at 60 seconds after starting.

(I.C.E., February, 1903.)

37. State and explain fully Newton's third law of motion. A 100-lb. shot leaves a gun horizontally with a muzzle velocity of 2000 feet per second. The gun and attachments, which recoil, weigh 4 tons. Find what the resistance must be that the recoil may be taken up in 4 feet, and compare the energy of recoil with the energy of translation of the shot. (I.C.E., February, 1903.)

38. An elastic string is used to lift a weight of 20 lbs. How much energy must be exerted in raising it 3 feet, supposing the string to stretch 1 inch under a tension of 1 lb.? Represent it graphically. If the work of stretching the string is lost, what is the efficiency of this method of lifting? (I.C.E., February, 1903.)

39. Explain how to determine graphically the relative velocity of two points the magnitudes and directions of whose velocities are known. Find the true course and velocity of a steamer steering due north by compass at 12 knots, through a 4-knot current setting south-west, and determine the alteration of direction by compass in order that the steamer should make a true northerly course.

(I.C.E., October, 1902.)

40. Find, by graphic construction, the centre of gravity of a section of an I beam, top flange 4 inches by 1 inch; web, between flanges, 14 inches by $1\frac{1}{2}$ inches; bottom flange 9 inches by 2 inches.

(I.C.E., October, 1902.)

41. A crankshaft, diameter $12\frac{1}{2}$ inches, weighs 12 tons, and it is pressed against the bearings by a force of 36 tons horizontally. Find the horse-power lost in friction at 90 revolutions per minute (coefficient of friction = 0·06.) (I.C.E., October, 1902.)

Questions selected from the Board of Education Examinations in Applied Mechanics.

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1. A truck, weighing 5 tons without its wheels, rests on 4 wheels, which are circular discs. 40 inches in diameter, each weighing $\frac{1}{2}$ ton, and moves down an incline of 1 in 60. Find the velocity of the truck in feet per second after moving 100 feet from rest, if the resistance due to friction is 1 per cent. of the weight. What percentage of the original potential energy has been wasted in friction? (Stage 3, 1905.)

2. A flywheel is supported on an axle $2\frac{1}{2}$ inches in diameter, and is rotated by a cord, which is wound round the axle and carries a weight. It is found by experiment that a weight of 5 lbs. on the cord is just sufficient to overcome the friction and maintain steady motion. A load of 25 lbs. is attached to the cord, and 3 seconds after starting from rest it is found that the weight has descended 5 feet. Find the moment of inertia of this wheel in engineers' units.

If the wheel is a circular disc 3 feet in diameter, what is its weight? (The thickness of the cord may be neglected.)

(Stage 3, 1905.)

3. The angular position D of a rocking shaft at any time t is measured from a fixed position. Successive positions at intervals at $\frac{1}{50}$ second have been determined as follows :—

Time t, seconds	0·0	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16	0·18
Position D, radians	0·106	0·208	0·337	0·487	0·651	0·819	0·978	1·111	1·201	1·222

Find the change of angular position during the first interval from $t = 0$ to $t = 0·02$. Calculate the mean angular velocity during this interval in radians per second, and, on a time base, set this up as an ordinate at the middle of the interval. Repeat this for the

other intervals, tabulating the results and drawing the curve showing approximately angular velocity and time.

In the same way, find a curve showing angular acceleration and time. Read off the angular acceleration in radians per second per second, when $t = 0{\cdot}075$ second. (Stage 2, 1905.)

4. A motor car moves in a horizontal circle of 300 feet radius. The track makes sideways an angle of 10° with the horizontal plane. A plumb-line on the car makes an angle of 12° with what would be a vertical line on the car if it were at rest on a horizontal plane. What is the speed of the car? If the car is just not side-slipping, what is the coefficient of friction? (Stage 3, 1904.)

5. A body whose weight is 350 lbs. is being acted upon by a variable lifting force F lbs. when it is at the height x feet from its position of rest. The mechanism is such that F depends upon x in the following way; but the body will stop rising before the greatest x of the table is reached. Where will it stop?

x	0	15	25	50	70	100	125	150	180	210
F	530	525	516	490	425	300	210	160	110	90

Where does its velocity cease to increase and begin to diminish? (Stage 3, 1904.)

6. Part of a machine weighing 1 ton is moving northwards at 60 feet per second. At the end of 0·05 second it is found to be moving to the east at 20 feet per second. What is the average force (find magnitude and direction) acting upon it during the interval 0·05 second? What is meant by "average" in such a case? What is meant by *force* by people who have to make exact calculations? (Stage 3, 1904.)

7. A flywheel and its shaft weigh 24,000 lbs.; its bearings, which are slack, are 9 inches diameter. If the coefficient of friction is 0·07, how many foot-pounds of work are wasted in overcoming friction in one revolution?

If the mean radius (or rather the radius of gyration) is 10 feet, what is the kinetic energy when the speed is 75 revolutions per minute? If it is suddenly disconnected from its engine at this speed, in how many revolutions will it come to rest? What is its average speed in coming to rest? In how many minutes will it come to rest? (Stage 2, 1904.)

8. A train, weighing 250 tons, is moving at 40 miles per hour, and it is stopped in ten seconds. What is the average force during

these ten seconds causing this stoppage? Define what is meant by force by people who have to make exact calculations.

(Stage 2, 1904.)

9. A tram-car, weighing 15 tons, suddenly had the electric current cut off. At that instant its velocity was 16 miles per hour. Reckoning time from that instant, the following velocities, V, and times, t, were noted :—

V, miles per hour	16	14	12	10
t, seconds	0	9·3	21	35

Calculate the average value of the retarding force, and find the average value of the velocity from $t = 0$ to $t = 35$. Also find the distance travelled between these times. (Advanced, 1903.)

10. A projectile has kinetic energy = 1,670,000 foot-lbs. at a velocity of 3000 feet per second. Later on its velocity is only 2000 feet per second. How much kinetic energy has it lost? What is the cause of this loss of energy? Calculate the kinetic energy of rotation of the projectile if its weight is 12 lbs., and its radius of gyration is 0·75 inch, and its speed of rotation is 500 revolutions per second. (Advanced, 1903.)

11. A weight of 10 lbs. is hung from a spring, and thereby causes the spring to elongate to the extent of 0·42 foot. If the weight is made to oscillate vertically, find the time of a complete vibration (neglect the mass of the spring itself). (Advanced, 1903.)

12. A flywheel weighs 5 tons and has a radius of gyration of 6 feet. What is its moment of inertia? It is at the end of a shaft 10 feet long, the other end of which is fixed. It is found that a torque of 200,000 lb.-feet is sufficient to turn the wheel through 1°. The wheel is twisted slightly and then released: find the time of a complete vibration. How many vibrations per minute would it make? (Honours, Part I., 1903.)

13. A flywheel of a shearing machine has 150,000 foot-lbs. of kinetic energy stored in it when its speed is 250 revolutions per minute. What energy does it part with during a reduction of speed to 200 revolutions per minute?

If 82 per cent. of this energy given out is imparted to the shears during a stroke of 2 inches, what is the average force due to this on the blade of the shears? (Advanced, 1902.)

14. A weight of 5 lbs. is supported by a spring. The stiffness of the spring is such that putting on or taking off a weight of 1 lb. produces a downward or upward motion of 0·04 foot. What is the time of a complete oscillation, neglecting the mass of the spring?

(Advanced, 1902.)

15. A car weighs 10 tons: what is its mass in engineers' units? It is drawn by the pull P lbs., varying in the following way, t being seconds from the time of starting :—

P ...	1020	980	882	720	702	650	713	722	805
t ...	0	2	5	8	10	13	16	19	22

The retarding force of friction is constant and equal to 410 lbs. Plot P − 410 and the time t, and find the *time average* of this excess force. What does this represent when it is multiplied by 22 seconds? What is the speed of the car at the time 22 seconds from rest? (Advanced, 1902.)

16. A body weighing 1610 lbs. is lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F lbs. of the rope. When the body had been lifted x feet from its position of rest, the pulling force was automatically recorded as follows :—

x ...	0	11	20	34	45	55	66	76
F ...	4010	3915	3763	3532	3366	3208	3100	3007

Using squared paper, find the velocity v feet per second for values of x of 10, 30, 50, 70, and draw a curve showing the probable values of v for all values of x up to 80. In what time does the body get from $x = 45$ to $x = 55$? (Honours, Part I., 1901.)

17. A machine is found to have 300,000 foot-lbs. stored in it as kinetic energy when its main shaft makes 100 revolutions per minute. If the speed changes to 98 revolutions per minute, how much kinetic energy has it lost? A similar machine (that is, made to the same drawings, but on a different scale) is made of the same material, but with all its dimensions 20 per cent. greater. What will be its store of energy at 70 revolutions per minute? What energy will it store in changing from 70 to 71 revolutions per minute? (Honours, Part I., 1901.)

18. A body of 60 lbs. has a simple vibration, the total length of a swing being 3 feet; there are 200 *complete* vibrations (or double swings) per minute. Calculate the forces which act on the body at the ends of a swing, and show on a diagram to scale what force acts upon the body in every position. (Advanced, 1901.)

19. An electric tramcar, loaded with 52 passengers, weighs altogether 10 tons. On a level road it is travelling at a certain

speed. For the purpose of finding the tractive force, the electricity is suddenly turned off, and an instrument shows that there is a retardation in speed. How much will this be if the tractive force is 315 lbs.? If the tractive force is found on several trials to be, on the average—

342	lbs. when	the	speed is	12	miles	per hour
315	"	"	"	10	"	"
294	"	"	"	8	"	"

what is the probable tractive force at 9 miles per hour?

(Advanced, 1901.)

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
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·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine			
Degrees.	Radians.								
0°	0	000	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

INDEX

(The numbers refer to pages)

(*The numbers refer to pages*)

F

G

H

I

K

L

M

N

(*The numbers refer to pages*)

(*The numbers refer to pages*)

THE END

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